These considerations are exploratory and make no pretence at competing with current systematic many-body techniques. Still they provide a different outlook, and it is possible that in the course of time transformation groups will play a more important role in many-body theory.

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3

THE MICROSCOPIC DESCRIPTION OF SUPERFLUIDITY‡

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1

Introduction

In these lectures I shall discuss the microscopic view of the phenomenon of superfluidity. My aims will be twofold: first, to show how macroscopic superfluid concepts, such as the superfluid velocity and the normal mass density, can be defined microscopically; and second, to show the deep unity underlying the descriptions of the superfluidity of low temperature He^4 and the superfluidity of electrons in superconductors.

The most convenient language for these purposes is that of second quantization. In this formalism the operator $\psi(\mathbf{r})$ acting to the right on a state of an N particle system *removes* a particle from the space point $\mathbf{r}$, producing a state of an $N-1$ particle system. Similarly the adjoint operator $\psi^\dagger(\mathbf{r})$, acting to the right, *adds* a particle to the state, at point $\mathbf{r}$. The creation and annihilation operators obey the commutation relation

$$\psi(\mathbf{r})\psi^\dagger(\mathbf{r}')\mp\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}') \qquad \text{...[1]}$$

where the upper sign is for bosons and the lower for fermions (always).

Suppose that $\phi_0(\mathbf{r}), \phi_1(\mathbf{r}), \ldots$ form a complete orthonormal set of single particle states. It is occasionally convenient to expand $\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r})$ in terms of the operators a_i and $a_i^\dagger$ that remove and add particles with the wave function $\phi_i(\mathbf{r})$. These expansions are

$$\psi(\mathbf{r}) = \sum_i \phi_i(\mathbf{r})a_i, \qquad \psi^\dagger(\mathbf{r}) = \sum_i \phi_i^*(\mathbf{r})a_i^\dagger. \qquad \text{...[2]}$$

The operators a_i and $a_i^\dagger$ obey

$$a_i a_j^\dagger \mp a_j^\dagger a_i = \delta_{ij}. \qquad \text{...[3]}$$

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A particularly useful case is when the ϕ_i are plane wave states (in a box of volume V with periodic boundary conditions). Then [2] becomes

$$\psi(r) = V^{-\frac{1}{2}} \sum_p e^{i p \cdot r} a_p, \qquad \psi^\dagger(r) = V^{-\frac{1}{2}} \sum_p e^{-i p \cdot r} a_p^\dagger. \qquad \ldots[4]$$

I shall usually work in the Heisenberg representation. Recall that in this representation the states remain fixed in time while each operator, X, develops in time according to the equation of motion [$\hbar = 1$, always]:

$$i \frac{\partial X(t)}{\partial t} = [X(t), H(t)] \qquad \ldots[5]$$

where $H(t)$ is the Hamiltonian of the system. In the case that H is independent of time, then from [5],

$$X(t) = e^{iHt} X e^{-iHt}, \qquad \ldots[6]$$

where X shall always denote the operator at $t = 0$. As an example, for free particles

$$H = \sum_p \varepsilon_p a_p^\dagger a_p, \qquad \ldots[7]$$

where

$$\varepsilon_p = p^2/2m; \qquad \ldots[8]$$

then

$$a_p(t) = e^{-i\varepsilon_p t} a_p, \qquad \ldots[9]$$

and the time-dependent creation and annihilation operators are given by

$$\begin{aligned} \psi(rt) &= V^{-\frac{1}{2}} \sum_p e^{i(p \cdot r - \varepsilon_p t)} a_p, \\ \psi^\dagger(rt) &= V^{-\frac{1}{2}} \sum_p e^{-i(p \cdot r - \varepsilon_p t)} a_p^\dagger. \end{aligned} \qquad \ldots[10]$$

2

Condensates

The crucial feature of all superfluid systems is the existence of a condensate, that is, a single state, or mode, that is macroscopically occupied. To understand this in detail, let us first review some properties of the correlation functions of normal, i.e., non-superfluid systems. Consider the single-particle correlation function[1]

$$G^<(rt, r't') = \langle \psi^\dagger(r't') \psi(rt) \rangle, \qquad \ldots[11]$$

where the expectation value is in the grand canonical ensemble:

$$\langle X \rangle = \frac{\sum_{E,N} e^{-\beta(E-\mu N)} \langle EN | X | EN \rangle}{\sum_{E,N} e^{-\beta(E-\mu N)}} = \frac{\mathrm{tr}\,[e^{-\beta(H-\mu N)} X]}{\mathrm{tr}\,[e^{-\beta(H-\mu N)}]}; \qquad \ldots[12]$$

$\beta = 1/KT$ and μ is the chemical potential. The function $G^<(rt, r't')$ is essentially the amplitude for removing a particle from "equilibrium" at r, t and returning to equilibrium by replacing a particle at r', t'.‡

For normal systems, the amplitude $G^<(rt, r't')$ falls to zero as the space-time point r', t' becomes widely separated from the point r, t. A few examples will illustrate this. For free particles, we find from [10] that

$$G^<(rt, r't') = V^{-1} \sum_p e^{ip\cdot(r-r')} e^{-i\varepsilon_p(t-t')} \langle a_p^\dagger a_p \rangle, \qquad \ldots[13]$$

since $\langle a_{p'}^\dagger a_p \rangle$ vanishes unless $p' = p$, i.e., if one removes a particle of momentum p from a free particle state, one can return to the same state only by adding back a particle with the same momentum p. Now $\langle a_p^\dagger a_p \rangle$ is just the expectation value of the number of particles, N_p, of momentum p in the system; for a grand canonical ensemble

$$\langle a_p^\dagger a_p \rangle = N_p = [e^{\beta(\varepsilon_p - \mu)} \mp 1]^{-1}. \qquad \ldots[14]$$

Let us look at the behavior of $G^<(r0, r'0)$ as $|r - r'| \to \infty$. In a classical system $N_p = e^{-\beta(\varepsilon_p - \mu)}$ and so by a simple integration ($V^{-1}\sum_p \to (2\pi)^{-3} d^3p$):

$$\begin{aligned} G^<(r0, r'0) &= (2\pi)^{-3} \int d^3p \, e^{ip\cdot(r-r') - \beta p^2/2m} e^{\beta\mu} \\ &= n e^{-m|r-r'|^2/2\beta}, \end{aligned} \qquad \ldots[15]$$

where n is the average particle density. Thus $G^<$ falls off as a Gaussian with a characteristic length equal to the thermal wavelength, $\lambda_{th} = (\hbar/mKT)^{\frac{1}{2}}$.

Similarly for $T = 0$ fermions, where

$$N_p = \begin{cases} 1, & p \leqslant p_f, \\ 0, & p > p_f, \end{cases} \qquad \ldots[16]$$

‡ The equal-time correlation function $G^<(r0, r'0)$ is proportional to the one-particle reduced density matrix. To see this note that a state $|EN\rangle$ with the properly symmetrized wave function $\Psi_{E,N}(r_1, \ldots r_N)$ can be constructed in second quantization as

$$|EN\rangle = (N!)^{-\frac{1}{2}} \int dr_1 \ldots dr_N \Psi_{E,N}(r_1, \ldots r_N) \psi^\dagger(r_1) \ldots \psi^\dagger(r_N) |\mathrm{vac}\rangle,$$

where $|\mathrm{vac}\rangle$ denotes the vacuum. Then

$$\psi(r)|EN\rangle = [(N-1)!]^{-\frac{1}{2}} \int dr_2 \ldots dr_N \Psi_{E,N}(r, r_2, \ldots r_N) \psi^\dagger(r_2) \ldots \psi^\dagger(r_N) |\mathrm{vac}\rangle,$$

and by repeated use of the commutation relations one finds

$$\langle EN | \psi^\dagger(r')\psi(r) | EN \rangle = N \int dr_2 \ldots dr_N \Psi_{E,N}^*(r', r_2, \ldots, r_N) \Psi_{E,N}(r, r_2, \ldots r_N).$$

Thus $G^<(r0, r'0)$ is simply the one-particle reduced density matrix, times the volume of the system.

we find

$$G^{<}(r0, r'0) = 3nx^{-3} (\sin x - x \cos x) \qquad ...[17]$$

where $x = p_f |r-r'|$. Again $G^{<}$ falls to zero for $|r-r'|$ large compared with the Fermi wavelength, $\hbar/p_f$.

It is also easy to verify for these two examples that $G^{<}(rt, r't')$ vanishes as $|t-t'| \to \infty$. In fact, it is a general property of normal systems that $G^{<}(rt, r't') \to 0$, as $|r-r'|$ or $|t-t'| \to \infty$. The correlation function

$$G^{>}(rt, r't') \equiv \langle \psi(rt)\psi^{\dagger}(r't')\rangle \qquad ...[18]$$

exhibits the same behavior.

Now let us consider the free Bose gas. At $T = 0$, all the particles are in the lowest energy state of the box; this state is macroscopically occupied. Then

$$N_p = \begin{cases} N, & p = 0 \\ 0, & p \neq 0 \end{cases} \qquad ...[19]$$

and we find from [13]

$$G^{<}(rt, r't') = N/V. \qquad ...[20]$$

In contrast to the previous examples, $G^{<}$ remains *constant* as r', t' becomes separated from r, t. For $T < T_c$, the Bose-Einstein condensation temperature, the ground state remains macroscopically occupied, $\mu = 0$, and

$$G^{<}(rt, r't') = n_0 + \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(r-r') - i\epsilon_p(t-t')}}{e^{\beta\epsilon_p}-1}. \qquad ...[21]$$

The second term vanishes as $|r-r'|$ or $|t-t'| \to \infty$. However the $n_0 = N_0/V$ term remains non-zero.

What is the difference between this and the previous cases? In a normal system the "hole" that one creates by removing a particle at r, t diffuses out (like the wave packet of a localized particle) and it becomes less and less probable that one will be able to return to the original state by adding a particle at r', t'. On the other hand in the condensed Bose gas the following possibility can occur: the particle removed at r, t can come from the condensate, the $p = 0$ mode; the relative amplitude for this possibility is $(n_0)^{\frac{1}{2}}$. Then when a particle is added at r', t' it has amplitude $(n_0)^{\frac{1}{2}}$ to fall into the condensate, returning the system to its original state. The amplitude for these two successive possibilities is thus $(n_0^{\frac{1}{2}})^2 = n_0$, *independent* of the separation of r from r' or t from t'. As long as the $p = 0$ state contains a macroscopic number of particles, this possibility has a finite probability amplitude.

If the Bose gas is in some external potential and $\phi_i(rt)$ denotes a single-

particle eigenstate of the potential, where $\phi_0(rt)$ is the macroscopically occupied ground state, then

$$\begin{aligned} G^{<}(rt, r't') &= \sum_i N_i\phi_i(rt)\phi_i^*(r't') \\ &\equiv N_0\phi_0(rt)\phi_0^*(r't') + \tilde{G}^{<}(rt, r't'), \qquad ...[22] \\ G^{>}(rt, r't') &= N_0\phi_0(rt)\phi_0^*(r't') + \tilde{G}^{>}(rt, r't'). \end{aligned}$$

The condensate contribution, the N_0 term, remains finite as r',t' becomes separated from r,t; on the other hand, the non-condensate contribution $\tilde{G}^{\gtrless}$ has the same behavior as in a normal system. Note that $N_0 |\phi_0(rt)|^2$ is the local density of condensate particles.

Now, as was originally proposed by Onsager and Penrose, this same structure persists in an interacting superfluid Bose system that is in, or not too far from, equilibrium; there is a single-particle state, characterized by a wave function $\Psi(rt)$, which is macroscopically occupied. The occupants of this state are the condensate and $\Psi(rt)$ is called the condensate wave function, or superfluid order parameter. It is defined by the relation

$$G^{\gtrless}(rt, r't') \to \Psi(rt)\Psi^*(r't'), \qquad ...[23]$$

as either $|r-r'|$ or $|t-t'|$ becomes very large. Then $|\Psi(rt)|^2$ is the density of particles in the condensate.

In a system at rest, the condensed mode is the $p = 0$ state and $\Psi(rt) = \sqrt{n_0}e^{-i\omega_0 t}$. The energy ω_0 is in fact the chemical potential μ of the system.‡ In real He^4 at $T = 0$ the density of particles in the condensate n_0 is estimated to be between 4% and 11% of the total density $n = 2{\cdot}2\times10^{22}\text{cm}^{-13}$. Consequently $n_0 \sim 10^{21}\text{cm}^{-3}$, which is quite macroscopic.

The single-particle correlation functions for a superfluid Bose system in or near equilibrium thus have the form

$$G^{\gtrless}(rt, r't') = \Psi(rt)\Psi^*(r't') + \tilde{G}^{\gtrless}(rt, r't'). \qquad ...[24]$$

We can think of [24] as arising in the following way. Imagine that $|\zeta, N\rangle$ is a "typical" state in the ensemble where N is the average number of particles. Then the condensate wave function, which is the relative amplitude for removing a particle at r, t from the condensate, is essentially given by

$$\Psi(rt) = \langle \zeta, N-1 | \psi(rt) | \zeta, N\rangle \qquad ...[25]$$

where $|\zeta, N-1\rangle$ is the same state as $|\zeta, N\rangle$, only with one less particle in the condensate. Similarly

‡ To see this recall that μ is the energy required to add a particle to the system at constant entropy. The occupied condensed mode is a zero entropy configuration and thus adding a particle to the condensate does not change the system's entropy; hence the energy ω_0 required to add a particle to the condensate must be μ. In an interacting Bose system μ is no longer zero; for example in liquid He^4 at $T = 0$, $\mu = -7{\cdot}16$°K.

$$\Psi^*(rt) = \langle \zeta, N | \psi^\dagger(rt) | \zeta, N-1 \rangle. \qquad \ldots[26]$$

Now $|\zeta, N\rangle$ and $|\zeta, N \pm 1\rangle$ are physically the same states; the one extra or fewer particles in the condensate can lead to corrections only of order one over the total number of particles in the condensate. For this reason one often writes

$$\Psi(rt) = \langle \psi(rt) \rangle \qquad \ldots[27]$$

with the understanding that the states on the left have one fewer condensate particle than the states on the right. Similarly

$$\Psi^*(rt) = \langle \psi^\dagger(rt) \rangle. \qquad \ldots[28]$$

One can then write the correlation function $G^<$ as

$$G^<(rt, r't') = \sum_{\zeta'} \langle \zeta, N | \psi^\dagger(r't') | \zeta', N-1 \rangle \langle \zeta', N-1 | \psi(rt) | \zeta, N \rangle. \qquad \ldots[29]$$

Then comparing with [24], we see that $\tilde{G}^<$ is the sum in [29], excluding the term $\zeta' = \zeta$.

In the free Bose gas

$$\psi(rt) = V^{-\frac{1}{2}} a_0 + V^{-\frac{1}{2}} \sum_{p \neq 0} e^{ip\cdot r - i\epsilon_p t} a_p. \qquad \ldots[30]$$

The Bogoliubov method‡ for solving the weakly interacting Bose gas is to replace a_0 by the c-number $(N_0)^{\frac{1}{2}}$. In analogy it is convenient to think of the operator $\psi(rt)$ in general as the sum of two terms

$$\psi(rt) = \langle \psi(rt) \rangle + \tilde{\psi}(rt) \qquad \ldots[31]$$

where $\tilde{\psi}(rt)$ removes non-condensate particles. From [31], $\langle \tilde{\psi}(rt) \rangle = 0$ and we may write

$$G^<(rt, r't') = \langle \psi(rt) \rangle \langle \psi^\dagger(r't') \rangle + \langle \tilde{\psi}^\dagger(r't') \tilde{\psi}(rt) \rangle. \qquad \ldots[32]$$

The latter term $\tilde{G}^<$ describes the fluctuations of the condensate.§

The argument that the existence of a condensate leads to superfluidity is essentially the following. Consider a state of the free Bose gas in which the particles are all condensed into a state with a finite momentum p. Such a state, in which the wave function of the condensed mode is proportional to $e^{ip\cdot r}$, would be one in which there is a macroscopic motion with all the particles flowing with a velocity $v = p/m$. Similarly a condensate wave function in the interacting system of the form

$$\Psi(rt) = e^{imv_s \cdot r - i\omega t} |\Psi| \qquad \ldots[33]$$

‡ See A. Abrikosov *et al.*, Ref. 1.

§ One should not be perturbed at the appearance of expressions that seem to violate particle conservation; the condensate acts as a reservoir of particles and the "missing" particles are always to be found in the condensate.

corresponds to a state of uniform macroscopic motion with a flow velocity v_s. This velocity is the *superfluid velocity*. The fundamental identification of the superfluid velocity is thus $1/m$ times the spatial rate of change of the phase of the condensate wave function. Generally, if one writes

$$\Psi(rt) = e^{iS(rt)} [n_0(rt)]^{\frac{1}{2}} \qquad \ldots[34]$$

where S and n_0 are real, then the local superfluid velocity is (at least in situations where S and n_0 vary slowly in space):

$$v_s(rt) = \nabla S(rt)/m. \qquad \ldots[35]$$

It is important to realize that unlike in normal systems, one can have relative flow of the superfluid with respect to the walls of the container and still have *equilibrium*. The superfluid velocity is thus an additional macroscopic variable in a superfluid.‡

[The grand canonical ensemble, as in [12], has the unpleasant characteristic of including states with different superfluid velocities. Since the superfluid velocity is a macroscopic observable one should, in fact, in describing a state with a given temperature, chemical potential and superfluid velocity, choose only those states from the grand canonical ensemble with the given v_s. This point will become slightly less abstract later.[2]]

The reason that superfluid flow is stable is that it is a coherent motion of a macroscopic number of particles. To slow down the flow requires coupling to all (or most of) the particles simultaneously. In contrast to a system such as a normal metal where thermal excitations slow down a current one electron at a time, the excitations in a superfluid, the phonons, rotons and vortex motions in He^4, have vanishingly small probability, at slow superfluid velocities, of disturbing the superfluid flow.

While the phase of the condensate wave function determines the superfluid velocity, the condensate mass density mn_0 is *not* the same as the superfluid mass density ρ_s of the two fluid picture. For example, in He^4 at $T = 0$, $\rho_s = \rho = mn$ which is 10 or 20 times greater than mn_0. The reason is that due to the interactions among the particles, there is a finite probability for non-condensate particles to be carried along with the condensate in the superfluid flow.

Superconductors also have condensates, formed of the Cooper pairs of electrons.[3] Recall Cooper's model of two electrons of opposite spin, outside a filled Fermi sea and interacting just with each other via an attractive interaction; the important feature of the model is that the electrons have a "bound state" of total energy less than twice the Fermi energy. In a superconductor, where all the electrons are interacting, there is a similar lowering of the energy from that of the normal state due to a "pairing" of electrons

‡ Technically, states with non-zero v_s (measured with respect to the walls) are only metastable; they have, however, an astronomical lifetime except for temperatures *very* close to the λ transition.

of opposite spin on opposite sides of the Fermi surface. The collection of pairs forms the condensate, in the sense that there is a finite amplitude for adding two opposite-spin particles and having them pair. Mathematically, the correlation function

$$G_2^<(r_1t_1, r_2t_2; r_1't_1', r_2't_2') = \langle \psi_\uparrow^\dagger(r_2't_2')\psi_\downarrow^\dagger(r_1't_1')\psi_\downarrow(r_1t_1)\psi_\uparrow(r_2t_2)\rangle \qquad ...[36]$$

(the arrows denote spin orientation) in a superconductor does not vanish as the primed variables become separated from the unprimed variables, as in a normal system; rather

$$G_2^<(r_1t_1, r_2t_2; r_1't_1', r_2't_2') \to f^*(r_1't_1', r_2't_2')f(r_1t_1, r_2t_2). \qquad ...[37]$$

The function $f(r_1t_1, r_2t_2)$ plays the role of the pair wave function; in analogy with [25] we can think of f as

$$f(rt, r't') = \langle \zeta, N-2 \mid \psi_\downarrow(rt)\psi_\uparrow(r't') \mid \zeta, N\rangle \qquad ...[38]$$

where $|\zeta,N\rangle$ is a typical state of the ensemble and $|\zeta, N-2\rangle$ is the same state, only with one fewer pair. Alternatively one can write

$$f(rt, r't') = \langle \psi_\downarrow(rt)\psi_\uparrow(r't')\rangle. \qquad ...[39]$$

The physics behind [37] is the same as in a Bose system [cf. Eq. [23]]; $G_2^<$ is the amplitude for removing two opposite-spin particles at r_1,t_1 and r_2,t_2 and then returning to equilibrium by adding two opposite-spin particles at r_1',t_1' and r_2',t_2'. In a superconductor there is a finite amplitude $f(r_1t_1, r_2t_2$ that the two removed particles come from the condensate and an amplitude $f^*(r_1't_1', r_2't_2')$ that the added particles fall into the condensate. The total amplitude for this process doesn't fall to zero as the primed variables become separated from the unprimed variables.

It is tempting to think of the condensate as being formed by a Bose condensation of two-electron "molecules". This picture isn't really valid for the reason that the size of the pairs, measured by the spatial extent of $f(rt, r't)$, is hundreds of interparticle spacings; thus there is tremendous spatial overlapping of the pairs.

It is important to realize that in a Bose system the condensate is formed purely kinematically—even a non-interacting Bose gas has a condensate—while in a superconductor the existence of the condensate is due to the dynamics of the system. This is a prime reason why the theory of superconductivity remained so elusive.

We define the condensate wave function for a superconductor by

$$\Psi(rt) = f(rt, rt) = \langle \psi_\downarrow(rt)\psi_\uparrow(rt)\rangle. \qquad ...[39a]$$

In a superconductor carrying no current the total momentum per pair is zero and

$$\Psi(rt) = e^{-2i\mu t} \mid \Psi(r) \mid \qquad ...[40]$$

(the energy required to remove a pair is twice the chemical potential of the electrons). In a state with a condensate wave function proportional to $e^{2iq\cdot r}$ each pair has total momentum $2q$; a superconductor in this state carries a supercurrent proportional to q‡.

Thus underlying the superfluidity of both He^4 and electrons in a superconductor is the existence of a condensate. In both systems there exist equilibrium states of superfluid flow, characterized by a condensate wave function whose phase varies in space.[4]

3

STATISTICAL MECHANICS OF MOVING SYSTEMS

One striking property of liquid He^4 is that if one begins to rotate a bucket of He^4 at $T = 0$ very slowly, then the fluid doesn't rotate with the bucket but rather remains at rest with respect to the laboratory; at finite temperatures, the moment of inertia of the superfluid is well reduced from the classical value. Similarly, the ability of a superfluid to flow through a channel without friction can be alternatively regarded as the inability of the channel, in moving past the fluid, to carry the fluid along with it. In order to study these effects microscopically, we should first review how one describes the statistical mechanics of systems with moving walls.

The answer is really very simple; the probability of an energy state occurring in the grand canonical ensemble is proportional to $e^{-\beta(E'-\mu N)}$ where E' is the energy of the system measured in the frame in which the walls are at *rest*, μ is the chemical potential in that frame, and N is the number of particles in the state. To see this, note that the probability of a given state occurring is the same to both an observer sitting in the laboratory and one moving with the walls. But in the frame in which the walls are at rest, the probability is proportional to $e^{-\beta(E'-\mu N)}$. Recall that never in a derivation of the Gibb's ensemble[5] need one ask if the walls define an inertial frame; all one uses is that they are at rest.

The problem is to find H', the Hamiltonian of the system in the frame in which the walls are at rest. We are interested in two cases: translation of the walls and rotation of the walls. The translating case is very simple; consider first a single particle, for which $H = p^2/2m$. Then the particle's velocity in the laboratory frame is p/m. In a frame moving with velocity u (the primed frame) the velocity is

$$v' = v - u. \qquad ...[41]$$

‡ One can think of the electrons as moving with an average velocity q/m. However in real superconductors, which are not translationally invariant, the use of the bare mass m to define a velocity is somewhat arbitrary; one might equally well use the band mass m^*. The physically important quantity is the momentum per pair, the gradient of the phase of the condensate wave function, and this does not depend on the choice of mass. See J. Bardeen, Ref. 4.

Thus the momentum in the primed frame is

$$p' = p - mu \qquad \ldots[42]$$

and

$$H' = \frac{p'^2}{2m} = \frac{p^2}{2m} - p\cdot u + \tfrac{1}{2}mu^2. \qquad \ldots[43]$$

This equation gives the Hamiltonian in the moving frame in terms of the coordinates of the laboratory frame. For a many-particle system the momentum of the particle i is $p_i' = p_i - m_i u$ so that

$$H' = \sum_i \frac{p_i'^2}{2m_i} + \tfrac{1}{2}\sum_{i,j} v(r_i' - r_j')$$

$$= \sum_i \frac{p_i^2}{2m_i} + \tfrac{1}{2}\sum_{i,j} v(r_i - r_j) - \sum_i p_i\cdot u + \tfrac{1}{2}\sum_i m_i u^2,$$

since coordinate differences are the same in both frames. Writing the total momentum as

$$P = \sum_i p_i \qquad \ldots[44]$$

and total mass as

$$M = \sum_i m_i \qquad \ldots[45]$$

we have

$$H' = H - P\cdot u + \tfrac{1}{2}Mu^2 \qquad \ldots[46]$$

and

$$P' = \sum_i p_i' = P - Mu. \qquad \ldots[47]$$

The transformation to a uniformly translating coordinate system is a Galilean transformation.

One can derive these transformation laws in a very formal fashion using second quantization. The question is: how is $\psi'(r')$, the annihilation operator in the primed frame, related to $\psi(r)$ in the unprimed frame? From [4] the field operator in the primed frame can be written as

$$\psi'(r') = V^{-\frac{1}{2}} \sum_p a_p' e^{ip\cdot r'}. \qquad \ldots[48]$$

Now removing a particle of momentum p in the primed frame is the same as removing one of momentum $p+mu$ in the unprimed frame. Thus $a_p' = a_{p+mu}$ and

$$\psi'(r') = V^{-\frac{1}{2}} \sum_p a_p e^{i(p-mu)\cdot r'} = e^{-imu\cdot r'}\psi(r).$$

At $t = 0$, $r' = r$ and we may write

$$\psi'(r) = e^{-imu\cdot r}\psi(r). \qquad \ldots[49]$$

Similarly

$$\psi'^{\dagger}(r) = e^{imu\cdot r}\psi^{\dagger}(r). \qquad \ldots[50]$$

It is left as an exercise to derive [46] and [47] by writing H' and P', which depend on ψ' and $\psi'^{\dagger}$, in terms of ψ and $\psi^{\dagger}$.

To discover the Hamiltonian in a rotating frame it is most straightforward to use the Lagrangian. For a single particle one has

$$v' = v - u, \qquad \ldots[51]$$

where now

$$u = \omega\times r = \omega\times r';$$

ω is the angular velocity of the rotating frame and primes refer to quantities measured in the rotating frame. The Lagrangians $\mathscr{L}$ and $\mathscr{L}'$ in the two frames must be equal (to within possible total derivatives with respect to time). Thus

$$\mathscr{L}'(v') = \mathscr{L}(v) = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(v'+u)^2.$$

Hence

$$p' = \frac{\partial \mathscr{L}'}{\partial v'} = m(v'+u) = mv = p; \qquad \ldots[52]$$

the momentum of a particle is the same in both the fixed and rotating frames. Then

$$H' = -\mathscr{L}' + p'\cdot v' = \frac{p^2}{2m} - p\cdot u. \qquad \ldots[53]$$

Since $p\cdot u = \omega\cdot(r\times p) = \omega\cdot L$, where L is the angular momentum, we have the result

$$H' = H - \omega\cdot L. \qquad \ldots[54]$$

This equation is also valid for a system of many particles interacting through central forces; L is then the total angular momentum.

We conclude that to describe a system whose walls are moving linearly with velocity u, the relative probability of a state of energy E and momentum P is

$$\text{probability} \propto e^{-\beta[E - P\cdot u + (\frac{1}{2}mu^2 - \mu)N]}. \qquad \ldots[55]$$

When the walls are rotating with angular velocity ω we have instead

$$\text{probability} \propto e^{-\beta[E - \omega\cdot L - \mu N]}. \qquad \ldots[56]$$

At this point I would like to digress for a moment on the transformation properties of elementary excitations. As is well known, Landau[5] pictures the low-lying excited states of superfluid He^4 as being composed of several (non-interacting) elementary excitations, the phonons and rotons. In the frame in which $\boldsymbol{v}_s = \mathbf{0}$ an excitation of momentum $\boldsymbol{p}$ has an energy ε_p; for small p, $\varepsilon_p = sp$, where s is the first sound velocity. Thus in the frame in which $\boldsymbol{v}_s = \mathbf{0}$ the total energy and momentum are given by

$$E = E_0 + \sum_p N_p \varepsilon_p \qquad \ldots[57]$$

$$\boldsymbol{P} = \boldsymbol{P}_0 + \sum_p N_p \boldsymbol{p} \qquad \ldots[58]$$

where E_0 and $\boldsymbol{P}_0$ are the energy and momentum (actually zero) of the ground state, and N_p is the number of excitations present of momentum $\boldsymbol{p}$. In thermal equilibrium with $\boldsymbol{v}_s = \mathbf{0}$ and the walls at rest

$$\langle N_p \rangle = [e^{\beta \varepsilon_p} - 1]^{-1}. \qquad \ldots[59]$$

Query: What is the energy of an excitation in a frame in which $\boldsymbol{v}_s \neq \mathbf{0}$? Then, we apply the results [46] and [47] with $\boldsymbol{u} = -\boldsymbol{v}_s$ to find

$$\boldsymbol{P}' = \boldsymbol{P} + M\boldsymbol{v}_s = (\boldsymbol{P}_0 + M\boldsymbol{v}_s) + \sum_p N_p \boldsymbol{p} \qquad \ldots[60]$$

and

$$E' = E_0 + \sum_p N_p \varepsilon_p + \boldsymbol{P}\cdot\boldsymbol{v}_s + \tfrac{1}{2}Mv_s^2$$

$$= E_0' + \sum_p N_p(\varepsilon_p + \boldsymbol{p}\cdot\boldsymbol{v}_s) \qquad \ldots[61]$$

where $E_0' = E_0 + \boldsymbol{P}_0\cdot\boldsymbol{v}_s + \tfrac{1}{2}Mv_s^2$. From [60] and [61] we can draw two conclusions. First, from [60], increasing N_p (the number in the $\boldsymbol{v}_s = \mathbf{0}$ frame) by 1 increases $\boldsymbol{P}'$ by $\boldsymbol{p}$; thus the momentum of an excitation in the primed frame is the same as in the frame with $\boldsymbol{v}_s = \mathbf{0}$:

$$\boldsymbol{p}' = \boldsymbol{p}. \qquad \ldots[62]$$

However, increasing N_p by 1 increases E' by $\varepsilon_p + \boldsymbol{p}\cdot\boldsymbol{v}_s$; thus the energy of an excitation in the frame with $\boldsymbol{v}_s \neq \mathbf{0}$ is‡

$$\varepsilon_p' = \varepsilon_p + \boldsymbol{p}\cdot\boldsymbol{v}_s. \qquad \ldots[63]$$

‡ Hidden in this analysis is the assumption that the energy of the excitations do not depend on the velocity of the walls; for by simply making a Galilean transformation to generate a superfluid velocity, we also change the velocity of the walls by $\boldsymbol{v}_s$. This assumption, which is valid for slow relative motion of the superfluid and walls, would not be valid if, for example, the relative motion of the superfluid and walls generated excitations which in turn interacted with the excitation whose energy we are examining.

In thermal equilibrium

$$\langle N_p \rangle = [e^{\beta(\varepsilon_p + \boldsymbol{p}\cdot(\boldsymbol{v}_s - \boldsymbol{v}_n))} - 1]^{-1}. \qquad \ldots[64]$$

where $\boldsymbol{v}_n$ is the velocity of the walls and $\varepsilon_p + \boldsymbol{p}\cdot(\boldsymbol{v}_s - \boldsymbol{v}_n)$ is the excitation energy measured in the frame in which the walls are at rest.

It is instructive to contrast the results [62] and [63] with the behavior of a slowly moving impurity, such as a He^3 atom, in He^4. In the frame where $\boldsymbol{v}_s = \mathbf{0}$ an impurity excitation of momentum $\boldsymbol{p}$ has an energy of the form

$$\varepsilon_p = \varepsilon_0 + p^2/2m^*, \qquad \ldots[65]$$

where m^* is some effective mass greater than the bare mass, m_i, of the impurity. From [65] the velocity of the impurity is

$$\boldsymbol{v} = \nabla_p \varepsilon_p = \boldsymbol{p}/m^*; \qquad \ldots[66]$$

the momentum of the excitation is therefore

$$\boldsymbol{p} = m^*\boldsymbol{v} \equiv m_i\boldsymbol{v} + \delta m^*\boldsymbol{v}. \qquad \ldots[67]$$

The $m_i\boldsymbol{v}$ is the momentum of the particle of mass m_i travelling at velocity $\boldsymbol{v}$. As the particle moves it causes a flow pattern in the He^4, as in Fig. 1, which carries a momentum $\delta m^*\boldsymbol{v}$; this is the second term in [67]. The total momentum of the excitation is thus that carried by the particle plus that carried by the He^4.

Assuming the He^4 without the impurity to be in its ground state, the energy and momentum of the He^4 plus impurity in the frame where $\boldsymbol{v}_s = \mathbf{0}$ are

$$E = E_0 + \varepsilon_p \qquad \ldots[68]$$

$$\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{p}. \qquad \ldots[69]$$

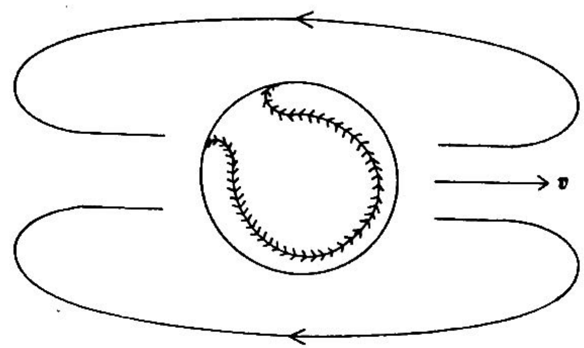

FIG. 1. Flow pattern induced by a slowly moving hard sphere in a fluid.

Then in the frame where $\boldsymbol{v}_s \neq \boldsymbol{0}$ one has

$$\begin{aligned} E' &= E+\boldsymbol{P}\cdot\boldsymbol{v}_s+\tfrac{1}{2}(M+m_i)v_s^2 \\ &= E_0'+(\varepsilon_{\boldsymbol{p}}+\boldsymbol{p}\cdot\boldsymbol{v}_s+\tfrac{1}{2}m_i v_s^2); \end{aligned} \quad \ldots[70]$$

as before $E_0' = E_0+\boldsymbol{P}_0\cdot\boldsymbol{v}_s+\frac{1}{2}Mv_s^2$. The tricky point here is that the impurity adds a mass m_i to the system, so that the total mass in the presence of the impurity is $M+m_i$. Similarly,

$$\boldsymbol{P}' = \boldsymbol{P}_0'+(\boldsymbol{p}+m_i\boldsymbol{v}_s). \quad \ldots[71]$$

Thus in the frame where $\boldsymbol{v}_s \neq \boldsymbol{0}$ the impurity excitation has a momentum

$$\boldsymbol{p}' = \boldsymbol{p}+m_i\boldsymbol{v}_s$$

and an energy

$$\varepsilon' = \varepsilon_{\boldsymbol{p}}+\boldsymbol{p}\cdot\boldsymbol{v}_s+\tfrac{1}{2}m_i v_s^2. \quad \ldots[72]$$

An impurity excitation of momentum $\boldsymbol{p}$ in the frame where $\boldsymbol{v}_s \neq \boldsymbol{0}$ therefore has an energy

$$\varepsilon_{\boldsymbol{p}}' = \varepsilon_{\boldsymbol{p}}+\frac{\delta m^*}{m^*}\,\boldsymbol{p}\cdot\boldsymbol{v}_s-\frac{1}{2}\,\frac{\delta m^*}{m^*}\,m_i v_s^2, \quad \ldots[73]$$

a result quite different from [63]. The reason for this difference is that unlike Landau's excitations, the impurity adds a mass m_i to the system. For $m_i = 0$, [73] reduces to [63].

One can get some insight into [73] by calculating the velocity of the impurity in the frame where $\boldsymbol{v}_s \neq \boldsymbol{0}$:

$$\boldsymbol{v} = \nabla_{\boldsymbol{p}}\varepsilon_{\boldsymbol{p}}' = (\boldsymbol{p}+\delta m^*\boldsymbol{v}_s)/m^* \quad \ldots[74]$$

or

$$\boldsymbol{p} = m_i\boldsymbol{v}+\delta m^*(\boldsymbol{v}-\boldsymbol{v}_s). \quad \ldots[75]$$

This says simply that the extra momentum carried by the He^4 depends on the velocity measured with respect to the superfluid velocity; the reason is that when $\boldsymbol{v} = \boldsymbol{v}_s$ the particle is at rest with respect to the superfluid, it induces no flow pattern and hence the superfluid carries no extra momentum due to the particle.

Arguments similar to those for the impurity determine the transformation properties of particle and hole excitations in superconductors.

4

Normal Mass Density and Moment of Inertia of a Superfluid

Consider a very long cylindrical pipe, as in Fig. 2, containing a superfluid in equilibrium, flowing with superfluid velocity $\boldsymbol{v}_s$. From the point of view

of the two-fluid model, "in equilibrium" means that the normal component is at rest and in equilibrium with the walls of the pipe; the momentum of the fluid is carried entirely by the superfluid component. For small $\boldsymbol{v}_s$, the momentum density, $\boldsymbol{g}$, is related to $\boldsymbol{v}_s$ by

$$\boldsymbol{g} = \rho_s\boldsymbol{v}_s \quad \ldots[76]$$

where ρ_s is the superfluid mass density. For a pure superfluid at $T = 0$, $\rho_s = \rho = mn$, the total mass density of the fluid; ρ_s decreases with increasing temperature and becomes 0 when the fluid becomes normal. Our aim in this section is to calculate the two-fluid model parameter ρ_s microscopically, in terms of the properties of the equilibrium fluid at rest.

This is most readily done by transforming to the frame in which $\boldsymbol{v}_s = \boldsymbol{0}$; in this frame the walls move with velocity $\boldsymbol{u} = -\boldsymbol{v}_s$ and the momentum density of the fluid, according to [47] is

$$\boldsymbol{g} = \rho_s\boldsymbol{v}_s-\rho\boldsymbol{v}_s = \rho_n\boldsymbol{u}, \quad \ldots[77]$$

where the normal mass density is defined by

$$\rho_n = \rho-\rho_s. \quad \ldots[78]$$

Our method for identifying ρ_n, and hence ρ_s microscopically shall be to calculate the momentum carried by a fluid in which $\boldsymbol{v}_s = \boldsymbol{0}$ and the walls are moving with velocity $\boldsymbol{u}$, and compare with [77].

Using the result [55] for the density matrix in the presence of walls moving with velocity $\boldsymbol{u}$, we may write the expectation value of the momentum density operator $\boldsymbol{g}(\boldsymbol{r})$ as

$$\langle\boldsymbol{g}(\boldsymbol{r})\rangle_{\boldsymbol{u}} = \frac{\operatorname{tr} e^{-\beta[H-\boldsymbol{P}\cdot\boldsymbol{u}+\frac{1}{2}Mu^2-\mu N]}\boldsymbol{g}(\boldsymbol{r})}{\operatorname{tr} e^{-\beta[H-\boldsymbol{P}\cdot\boldsymbol{u}+\frac{1}{2}Mu^2-\mu N]}}; \quad \ldots[79]$$

the momentum density operator is m times the current density operator:

$$\boldsymbol{g}(\boldsymbol{r}) = m\boldsymbol{j}(\boldsymbol{r}) = \frac{1}{2i}\,[\psi^\dagger(\boldsymbol{r})\,\{\nabla\psi(\boldsymbol{r})\}-\{\nabla\psi^\dagger(\boldsymbol{r})\}\psi(\boldsymbol{r})]. \quad \ldots[80]$$

The total momentum is given by

$$\boldsymbol{P} = \int d\boldsymbol{r}\,\boldsymbol{g}(\boldsymbol{r}). \quad \ldots[81]$$

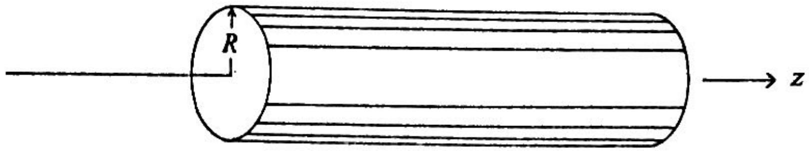

FIG. 2. A very long cylindrical pipe containing a superfluid in equilibrium, flowing with superfluid velocity $\boldsymbol{v}_s$. (See Section 4.)

We want to include only states with $\mathbf{v}_s = 0$ in the average in [79]; this is possible for the geometry of Fig. 2, where $\mathbf{u}$ is in the z direction, only if the pipe has "open ends", i.e. extends infinitely far in the positive and negative z directions.

To calculate ρ_n we need $\langle \mathbf{g}(\mathbf{r})\rangle_u$ to first order in $\mathbf{u}$. Expanding [79] to first order in $\mathbf{u}$ (taken along the z direction) we have

$$\langle g_z(\mathbf{r})\rangle_u = \langle g_z(\mathbf{r})\rangle + \beta[\langle g_z(\mathbf{r})P_z\rangle - \langle g_z(\mathbf{r})\rangle \langle P_z\rangle]u, \qquad \ldots[82]$$

where the expectation values on the right are in the equilibrium system at rest, $\mathbf{v}_s = 0$ and $\mathbf{u} = 0$; in this system $\langle \mathbf{g}(\mathbf{r})\rangle = 0$. (Because $[H, \mathbf{P}] = 0$ we needn't worry about the non-commutability of $g_z(\mathbf{r})$ and P_z in the *first* derivative with respect to $\mathbf{u}$ of the traces in [79]).

Before proceeding further it is necessary to develop some properties of correlation functions defined in thermal ensembles. Let $A(t)$ and $B(t')$ be operators, and consider the correlation function

$$\langle\{A(t)-\langle A(t)\rangle\}\,\{B(t')-\langle B(t')\rangle\}\rangle = \langle A(t)B(t')\rangle - \langle A(t)\rangle\,\langle B(t')\rangle.$$

From [12] and [6] we have

$$\begin{aligned}\langle A(t)B(t')\rangle &= \frac{1}{Z}\,\mathrm{tr}\{e^{-\beta(H-\mu N)}\,e^{iHt}A(0)\,e^{-iHt}B(t')\} \\ &= \frac{1}{Z}\,\mathrm{tr}\{[e^{-\beta(H-\mu N)}\,e^{iHt}A(0)\,e^{-iHt}\,e^{\beta(H-\mu N)}][e^{-\beta(H-\mu N)}B(t')]\}\end{aligned} \qquad \ldots[83]$$

where

$$Z = \mathrm{tr}\, e^{-\beta(H-\mu N)} \qquad \ldots[84]$$

is the partition function. Using the fact that $\mathrm{tr}\,XY = \mathrm{tr}\,YX$, [83] becomes

$$\begin{aligned}&\langle A(t)B(t')\rangle \\ &= \frac{1}{Z}\,\mathrm{tr}\{[e^{-\beta(H-\mu N)}B(t')][e^{\beta\mu N}\,e^{iH(t+i\beta)}A(0)\,e^{-iH(t+i\beta)}\,e^{-\beta\mu N}]\} \\ &= \langle B(t')\,e^{\beta\mu N}A(t+i\beta)\,e^{-\beta\mu N}\rangle,\end{aligned} \qquad \ldots[85]$$

where

$$A(t) \equiv e^{iHt}A(0)e^{-iHt} \qquad \ldots[86]$$

for complex, as well as real t. For the case that A commutes with N, [85] becomes

$$\langle A(t)B(t')\rangle = \langle B(t')A(t+i\beta)\rangle. \qquad \ldots[87]$$

In equilibrium $\langle A(t)B(t')\rangle$ depends only on the difference $t-t'$ and $\langle A(t)\rangle$ and $\langle B(t')\rangle$ are independent of time; introducing the Fourier transforms of the correlation functions,

$$\begin{aligned}Q(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}[\langle A(t)B(t')\rangle - \langle A\rangle\langle B\rangle] \\ R(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}[\langle B(t')A(t)\rangle - \langle B\rangle\langle A\rangle],\end{aligned} \qquad \ldots[88]$$

we find from [87] that

$$\begin{aligned}Q(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}[\langle B(t')A(t+i\beta)\rangle - \langle B\rangle\langle A\rangle] \\ &= \int_{-\infty}^{\infty} dt\, e^{i\omega(t-i\beta-t')}[\langle B(t')A(t)\rangle - \langle B\rangle\langle A\rangle]\end{aligned}$$

or

$$Q(\omega) = e^{\beta\omega}R(\omega). \qquad \ldots[89]$$

This relation is known as the "detailed balancing" condition.

Using [88] we can then write

$$Q(\omega) = \frac{\Upsilon_{AB}(\omega)}{1-e^{-\beta\omega}} \qquad \ldots[90]$$

where

$$\Upsilon_{AB}(\omega) \equiv Q(\omega) - R(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\langle[A(t), B(t')]\rangle \qquad \ldots[91]$$

Thus, we find that in the grand canonical ensemble, when $[A, N] = 0$,

$$\langle A(t)B(t')\rangle - \langle A\rangle\langle B\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{\Upsilon_{AB}(\omega)\,e^{-i\omega(t-t')}}{1-e^{-\beta\omega}}. \qquad \ldots[92]$$

A simple corollary of this theorem is that

$$\begin{aligned}\int_0^{-i\beta} d(t-t')(\langle A(t)B(t')\rangle - \langle A\rangle\langle B\rangle) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{\Upsilon_{AB}(\omega)}{1-e^{-\beta\omega}} \int_0^{-i\beta} dt\, e^{-i\omega t} \\ &= -i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{\Upsilon_{AB}(\omega)}{\omega}.\end{aligned} \qquad \ldots[93]$$

Since $\mathbf{P}$ is a constant of the motion $[\mathbf{P}(t) = \mathbf{P}(0)]$ we can write [82] as

$$\langle g_z(\mathbf{r})\rangle_u = i\int_0^{-i\beta} d(-t')\,[\langle g_z(\mathbf{r}0)P_z(t')\rangle - \langle g_z(\mathbf{r})\rangle\,\langle P_z\rangle]u, \qquad \ldots[94]$$

$$= i\int d\mathbf{r}'\int_0^{-i\beta} d(-t')\,[\langle g_z(\mathbf{r}0)g_z(\mathbf{r}'t')\rangle - \langle g_z(\mathbf{r})\rangle\,\langle g_z(\mathbf{r}')\rangle]u \qquad \ldots[95]$$

on using [81]. Then [93] implies

$$\langle g_z(\mathbf{r})\rangle_u = \int d\mathbf{r}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{zz}(\mathbf{r}, \mathbf{r}', \omega)}{\omega} u \qquad \ldots[96]$$

where

$$\Upsilon_{ij}(\mathbf{r}, \mathbf{r}', \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \langle [g_i(\mathbf{r}t), g_j(\mathbf{r}'t')]\rangle. \qquad \ldots[97]$$

For a translationally invariant system, this function depends only on $\mathbf{r}-\mathbf{r}'$, except when $\mathbf{r}$ and $\mathbf{r}'$ are near the boundaries of the system. To calculate ρ_n for a macroscopic system, these boundary regions can be neglected and we can replace Υ_{ij} by its value for an infinite system. Then

$$\langle g_z(\mathbf{r})\rangle = \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r}'\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{zz}(\mathbf{k}\omega)}{\omega} \qquad \ldots[98]$$

where $\Upsilon_{zz}(\mathbf{k}\omega)$ is the spatial Fourier transform of $\Upsilon_{zz}(\mathbf{r}, \mathbf{r}', \omega)$. The $\mathbf{r}'$ integration depends on the geometry of the system.

For the cylinder of Fig. 2, extending infinitely far in the positive and negative z directions, the z' integration gives $2\pi\delta(k_z)$; this delta function sets $k_z = 0$ in Υ_{zz}. For large R, the radius of the cylinder, the x' and y' integrals weight the integrand of [98] sharply around k_x and $k_y = 0$; in the limit of $R \to \infty$ the result of the x' and y' integrals is a factor $(2\pi)^2\delta(k_x)\delta(k_y)$. The important point to notice is that for the infinitely long cylinder, first we must set $k_z = 0$ and then let k_x and k_y approach zero. Hence

$$\langle g_z(\mathbf{r})\rangle_u = \lim_{k_x, k_y \to 0} \lim_{k_z \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{zz}(\mathbf{k}\omega)}{\omega} u; \qquad \ldots[99]$$

comparing this result with [77], we see that the coefficient of u on the right, which is independent of $\mathbf{r}$ for a very large system, equals ρ_n:

$$\rho_n = \lim_{k_x, k_y \to 0} \lim_{k_z \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{zz}(\mathbf{k}\omega)}{\omega}. \qquad \ldots[100]$$

This identification gives us a microscopic prescription for calculating the normal mass density in terms of the correlation function $\Upsilon_{zz}(\mathbf{k}\omega)$, which is evaluated in the equilibrium ensemble with the system entirely at rest. In a moment we shall simplify [100].

Before that let us consider a second situation. Suppose that the cylinder has closed ends at $z = \pm\frac{1}{2}L$. Then as one moves the walls with velocity $\mathbf{u}$ along the z direction one expects the *entire* fluid to follow, not just the normal component; it is physically impossible for the superfluid velocity to remain at rest when the closed container moves. With these boundary conditions, the states in [79] must have $\mathbf{v}_s = \mathbf{u}$. This experiment corresponds, in fact, simply to a Galilean transformation on the entire system, and the momentum density should be given by

$$\langle g_z\rangle_u = \rho u, \qquad \ldots[101]$$

where $\rho = mn$ is the total mass density of the system.

The calculation of $\langle g_z\rangle$ in this case parallels that just done, except that the closed ends require us to keep L finite while letting $R \to \infty$; finally we may let $L \to \infty$. The result is

$$\langle g_z(\mathbf{r})\rangle_u = \lim_{k_z \to 0} \lim_{k_x, k_y \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{zz}(\mathbf{k}\omega)}{\omega} u. \qquad \ldots[102]$$

Shortly we shall prove that the coefficient of u in [102] must equal ρ, as we argued above.

The limiting procedures in [100] and [102] can be carried out if we first write $\Upsilon_{ij}(\mathbf{k}\omega)$ as a sum of its transverse and longitudinal components. Since $\Upsilon_{ij}(\mathbf{k}\omega)$ transforms as a tensor, and $\mathbf{k}$ is the only vector on which it can depend, it must be a sum of a term proportional to δ_{ij} and a term proportional to k_ik_j/k^2. We write

$$\Upsilon_{ij}(\mathbf{k}\omega) = \frac{k_ik_j}{k^2}\Upsilon^L(k, \omega) + \left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\Upsilon^T(k, \omega). \qquad \ldots[103]$$

The first term is the longitudinal component; it is parallel to $\mathbf{k}$ in both indices, e.g., $\sum_i k_i(k_ik_j/k^2) = k_j$. The second term, the transverse component is perpendicular to $\mathbf{k}$ in both indices, e.g., $\sum_i k_i(\delta_{ij} - k_ik_j/k^2) = 0$.

Now observe that since $k^2 = k_x^2 + k_y^2 + k_z^2$,

$$\lim_{k_x, k_y \to 0} \lim_{k_z \to 0} \frac{k_zk_z}{k^2} = 0 \qquad \ldots[104]$$

while

$$\lim_{k_z \to 0} \lim_{k_x, k_y \to 0} \frac{k_zk_z}{k^2} = 1. \qquad \ldots[105]$$

Thus from [100] we find as the microscopic definition of ρ_n:

$$\rho_n = \lim_{k \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon^T(k, \omega)}{\omega}; \qquad \ldots[106]$$

furthermore the equality of [102] to ρu says that

$$\rho = \lim_{k \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon^L(k, \omega)}{\omega}. \qquad \ldots[107]$$

The fact that the transverse and longitudinal components of the current-current correlation function, as they enter [106] and [107], are different is a unique property of a superfluid. One can take as a microscopic criterion for superfluid behavior that ρ_n as calculated from [106] be less than ρ. We shall see later that exactly the same criterion applies to a superconductor; if $\rho_n < \rho$ then the system exhibits a Meissner effect.

The functions Υ^L and Υ^T are m^2 times the Fourier transforms of the correlation functions of the longitudinal component of the current with itself and the transverse component of the current with itself. Recall that any vector can be written as a sum of a curl-free, or longitudinal, part plus a divergence-free, or transverse, part. If $\mathbf{w}(\mathbf{r})$ is any vector function then

$$\mathbf{w}^L(\mathbf{r}) \equiv -\nabla \int d\mathbf{r}' \frac{\nabla' \cdot \mathbf{w}(\mathbf{r}')}{4\pi |\mathbf{r}-\mathbf{r}'|} \quad ...[108]$$

has no curl, while $\mathbf{w}^T(\mathbf{r}) = \mathbf{w}(\mathbf{r}) - \mathbf{w}^L(\mathbf{r})$ has zero divergence. In terms of the Fourier transform $\mathbf{w}(\mathbf{k})$ of $\mathbf{w}(\mathbf{r})$,

$$w_i^L(\mathbf{k}) = \sum_j \frac{k_i k_j}{k^2} w_j(\mathbf{k})$$

$$w_i^T(\mathbf{k}) = \sum_j \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) w_j(\mathbf{k}). \quad ...[109]$$

Now from [103]

$$\Upsilon^L(k, \omega) = \sum_{ij} \frac{k_i k_j}{k^2} \Upsilon_{ij}(k\omega);$$

the $k_i k_j / k^2$ picks out the longitudinal component g^L of the momentum density, so that we can write

$$\Upsilon^L(k, \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \langle [g_z^L(\mathbf{r}t), g_z^L(\mathbf{r}'t')] \rangle; \quad ...[110]$$

similarly

$$\Upsilon^T(k, \omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \langle [g_z^T(\mathbf{r}t), g_z^T(\mathbf{r}'t')] \rangle. \quad ...[111]$$

By symmetry the correlation function of g^T with g^L vanishes.

The longitudinal component of the current is that part associated with density changes. One sees this from the continuity equation

$$\frac{\partial \rho(\mathbf{r}t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r}t) = 0, \quad ...[112]$$

where $\rho(\mathbf{r}t)$ is the density; if the current has a longitudinal component then $\nabla \cdot \mathbf{j} \neq 0$ and so ρ changes in time. A transverse current satisfies $\nabla \cdot \mathbf{j}^T = 0$ and is therefore not accompanied by any density changes in time.

We turn now to the proof of [107]. The starting point is the expectation

value of the equal-time commutation relation between the density operator $\rho(\mathbf{r}t) = \psi^\dagger(\mathbf{r}t)\psi(\mathbf{r}t)$, and the current operator,

$$\langle [\rho(\mathbf{r}t), \nabla' \cdot \mathbf{j}(\mathbf{r}'t)] \rangle = i\nabla^2 \delta(\mathbf{r}-\mathbf{r}') n/m; \quad ...[113]$$

this is readily derived from [1]. Fourier transforming in space and time, [113] becomes

$$\mathbf{k} \cdot \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle [\rho, \mathbf{j}] \rangle (k\omega) = k^2 n/m \quad ...[114]$$

where $\langle [\rho, \mathbf{j}] \rangle (k\omega)$ denotes the Fourier transform of $\langle [\rho(\mathbf{r}t), \mathbf{j}(\mathbf{r}'t')] \rangle$. But from the continuity equation

$$m^2 \omega \langle [\rho, j_j] \rangle (k, \omega) = \sum_i k_i \Upsilon_{ij}(k\omega) = k_j \Upsilon^L(k, \omega). \quad ...[115]$$

Substituting [115] into [114] and cancelling the k^2 from both sides we derive

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon^L(k, \omega)}{\omega} = nm; \quad ...[116]$$

this relation is known as the longitudinal or f-sum rule. Eq. [107] is just the $k \to 0$ limit of [116].

The primary difference between the two experiments we have discussed is that in the first case, that of the cylinder with "open ends", the motion is carried out with $\mathbf{v}_s$ remaining zero, i.e., with the phase of the condensate wave function remaining constant in space. In the second case the superfluid, by virtue of the closed ends of the cylinder, must respond to the motion of the walls. Indeed, from [49] we see that the particle annihilation operator, ψ, in the laboratory frame is simply $e^{im\mathbf{u}\cdot\mathbf{r}}$ times ψ' in the system travelling with the walls. But in that frame $\langle \psi' \rangle = \Psi_0$, a constant. Therefore in the laboratory frame

$$\Psi(\mathbf{r}t) = \langle \psi(\mathbf{r}t) \rangle_u = e^{im\mathbf{u}\cdot\mathbf{r}} \Psi_0. \quad ...[117]$$

Thus the effect of a Galilean transformation is to change the phase of the condensate wave function by $m\mathbf{u}\cdot\mathbf{r}$: and hence $\mathbf{v}_s$ in general transforms into $\mathbf{v}_s + \mathbf{u}$, as it should, and in our example it goes from $\mathbf{0}$ to $\mathbf{u}$.

The difference between ρ_n and ρ is thus due to the response of the condensate to the motion of the walls. This can be made most explicit by returning to the frame in which the walls are stationary. In the situation of the cylinder with open ends the condensate has a velocity $\mathbf{v}_s = -\mathbf{u}$, and the momentum density [76] is $\rho_s \mathbf{v}_s$; for the closed cylinder, the momentum density and the condensate velocity are both zero.

Eq. [106] provides us with a starting point for calculating ρ_n and hence ρ_s microscopically. We have yet to show, however, that the existence of a condensate implies that ρ_s is non-zero; there does not exist a rigorous proof that this must always be so, but a little later we shall indicate the connection in a few simple situations.

We turn now to calculating the moment of inertia of a container full of superfluid. The moment of inertia tensor $\mathscr{I}_{ij}$ is defined, in terms of the expectation value of the angular momentum L of the system in a bucket rotating with angular velocity ω, by

$$\mathscr{I}_{ij} = \left[\frac{\partial \langle L_i \rangle_\omega}{\partial \omega_j}\right]_{\omega=0}. \qquad \ldots[118]$$

Using [56] we find, as in [82],

$$\mathscr{I}_{ij} = \beta[\langle L_i L_j \rangle - \langle L_i \rangle \langle L_j \rangle] \qquad \ldots[119]$$

where the expectation values are for the bucket and superfluid at rest in the laboratory. Now using the facts that

$$L = \int dr\, r \times g(r), \qquad \ldots[120]$$

and that L is a constant of the motion, and paralleling the calculation leading to [98], we find

$$\mathscr{I}_{ij} = \int \frac{dk}{(2\pi)^3} \int dr \int dr'\, e^{ik\cdot(r-r')} \varepsilon_{isl}\varepsilon_{jmn} r_s r'_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon_{ln}(k\omega)}{\omega}; \qquad \ldots[121]$$

ε_{ijk} is the completely anti-symmetric tensor, and summation over repeated indices is assumed. Let us consider a cylindrical bucket rotating about its axis, which we take to be along z, and calculate $\mathscr{I}_{zz}$, its moment of inertia about the z axis. From [103] Υ_{ln} contains terms proportional to δ_{ln} and to $k_l k_n/k^2$. This latter term in [121] leads to an r integral

$$\int dr e^{ik\cdot r}(r \times k)_z$$

which vanishes when the integration is symmetric about the z axis. In the δ_{ln} term we write the r' integral, in the limit of infinite volume, as

$$\begin{aligned}\int dr' e^{-ik\cdot r'} r'_m &= i(\nabla_k)_m \int dr' e^{ik\cdot r'} \\ &= i(\nabla_k)_m (2\pi)^3 \delta(k);\end{aligned} \qquad \ldots[122]$$

then integrating by parts with respect to k we have

$$\mathscr{I}_{zz} = i \int dr\ \varepsilon_{zsl}\varepsilon_{zml} r_s \left\{(\nabla_k)_m \left[e^{ik\cdot r} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Upsilon^T(k,\omega)}{\omega}\right]\right\}_{k=0}.$$

The ∇_k of Υ^T vanishes by symmetry as $k \to 0$. Then using the definition [106] of ρ_n we find that the moment of inertia of a very slowly rotating bucket is given by

$$\mathscr{I}_{zz} = \int dr\, (r^2 - z^2)\rho_n. \qquad \ldots[123]$$

This is the standard expression for the moment of inertia except that it is

proportional to the normal mass density, rather than ρ as it would be in a normal system. Thus in a superfluid the moment of inertia is reduced from its classical value. As the bucket is rotated the superfluid remains at rest in the laboratory, and only the normal mass density follows the bucket. At $T = 0$, ρ_n and hence $\mathscr{I}_{zz}$ vanishes; the entire fluid remains at rest as the bucket is rotated. It is the same ρ_n that determines the moment of inertia as determines the momentum density, $(\rho-\rho_n)v_s$, carried by a superfluid flow.

It is left as an exercise to show from [121] that the moment of inertia of a long cylinder of superfluid, as in Fig. 2, for rotations perpendicular to the symmetry axis, is given by the *classical* value.

Eq. [123] is valid as long as the bucket rotates slowly. Beyond some critical angular velocity ω_{c1} it becomes energetically favorable to create vortices in the fluid, which lead to an increase in the moment of inertia. ω_{c1} is given by $(\hbar/mR^2)\log(R/\xi)$, where ξ, the "coherence length", is of the order of a few Angstroms at low temperature; for $R = 1$ cm, $\omega_{c1} \sim 10^{-3}$ sec^{-1}. As ω is increased, $\mathscr{I}_{zz}$ keeps increasing until ω reaches a second critical value, ω_{c2}, where $\mathscr{I}_{zz}$ becomes the classical value, $\int dr(x^2+y^2)\rho$; beyond this point the fluid is normal.[6]

At very low temperatures, the source of the normal mass density is the thermal excitations – the phonons and rotons. The excitations come into thermal equilibrium with the walls and follow any motion of the walls. Landau's calculation of ρ_n illustrates this clearly. The momentum of the excitations is

$$\langle P \rangle = \sum p \langle N_p \rangle \qquad \ldots[124]$$

where in thermal equilibrium $\langle N_p \rangle$ is given by [64]; taking $v_s = 0$ and expanding to first order in v_n, the wall velocity, we have

$$\langle P \rangle = -\sum_p p(p\cdot v_n) \frac{\partial}{\partial \varepsilon_p} \frac{1}{e^{\beta\varepsilon_p}-1}, \qquad \ldots[125]$$

and thus

$$\rho_n = -\int \frac{dp}{(2\pi)^3} \frac{p^2}{3} \frac{\partial}{\partial \varepsilon_p} \frac{1}{e^{\beta\varepsilon_p}-1}. \qquad \ldots[126]$$

For $T \lesssim 0{\cdot}6$°K, $\rho_n \sim (T/T_D)^4$, where the phonon "Debye temperature" T_D is about 18°K. Incidentally, the same result for ρ_n emerges from a rather tedious microscopic calculation in the Bogoliubov approximation.

5

Linear Response Theory

In this section I shall develop the general formalism for describing the first-order response of a system in thermal equilibrium to an external perturbation. This formalism will enable us to study the criterion for a Meissner effect in a superconductor.

Consider applying a time-dependent perturbation, H' to a system initially in a state $|i, t_0\rangle$ at an early time. For example, in a charged system the perturbation might be an electromagnetic vector potential, for which

$$H' = \sum_i \frac{(p_i - eA(r_i)/c)^2}{2m} - \sum_i \frac{p_i^2}{2m}$$

$$= -\frac{e}{c}\int dr\, j(r)\cdot A(r) + \frac{e^2}{2mc^2}\int dr\, A^2(r)\rho(r). \qquad \ldots[127]$$

In the Heisenberg representation in which the operators change in time according to H, the system's Hamiltonian without H', the state of the system at a later time t is

$$|i, t\rangle = U(t, t_0)\,|i, t_0\rangle \qquad \ldots[128]$$

where (T denotes the time-ordered product):

$$U(t, t_0) = \mathrm{T}\left(\exp\left\{-i\int_{t_0}^{t} H'(t')dt'\right\}\right)$$

$$= 1 - i\int_{t_0}^{t} H'(t')dt' \qquad \ldots[129]$$

to first order in H'. The expectation value of an operator X at time t is then

$$\langle X(t)\rangle_{H'} = \langle i, t\,|\,X(t)\,|\,i, t\rangle$$
$$= \langle i, t_0\,|\,U^{-1}(t, t_0)X(t)U(t, t_0)\,|\,i, t_0\rangle \qquad \ldots[130]$$

where $X(t)$ develops in time according to H. For a system initially in thermal equilibrium we have

$$\langle X(t)\rangle_{H'} = \langle U^{-1}(t, t_0)X(t)U(t, t_0)\rangle, \qquad \ldots[131]$$

where the expectation value on the right is in a thermal ensemble.

To calculate the linear response of X to H' we use [129] and

$$U^{-1}(t, t_0) = 1 + i\int_{t_0}^{t} H'(t')dt'$$

to write

$$\langle X(t)\rangle_{H'} = \langle X(t)\rangle - i\int_{t_0}^{t} \langle [X(t), H'(t')]\rangle dt'. \qquad \ldots[132]$$

This is a general Kubo relation; it expresses the linear transport properties of a system in terms of expectation values (on the right) in the equilibrium ensemble. Note that [132] gives the retarded response of X to H'; $\langle X(t)\rangle_{H'}$ is determined by values of $H'(t')$ for times earlier than t.

As an example, let us calculate the electromagnetic current induced by a weak vector potential $A(rt)$ coupled to the system through [127]. Since the velocity of a particle of momentum p_i in the presence of A is

$$v_i = \frac{p_i - eA(r_i t)/c}{m}, \qquad \ldots[133]$$

the electromagnetic current operator, $\propto ev_i$, is

$$J(r) = ej(r) - \frac{e^2}{mc}A(rt)\rho(r), \qquad \ldots[134]$$

where $j(r)$, the "paramagnetic" current, is given by [80]. The second term on the right, the "diamagnetic" current is already first order in A, and so to calculate $\langle J\rangle$ to first order in A, we can take $\langle \rho(r)\rangle = n$. Thus

$$\langle J(rt)\rangle_A = e\langle j(rt)\rangle_A - \frac{ne^2}{mc}A(rt)$$

$$= \frac{ie^2}{c}\int_{-\infty}^{t} dt'\int dr'\langle [j(rt), j(r't')]\rangle\cdot A(r't') - \frac{ne^2}{mc}A(rt), \quad \ldots[135]$$

where we have used [132] for $\langle j(rt)\rangle$ and have let $t_0 \to -\infty$; also we assumed that $\langle j\rangle$ vanishes if $A = 0$. In a moment we shall use this result to discuss the Meissner effect in superconductors, but first let us see the connection between this approach and the calculations in the previous section.

Consider the response of a system to a perturbation

$$H'(t) = -m\int dr\;\; j(rt)\cdot u(rt). \qquad \ldots[136]$$

For $u(rt) = u(t)$, independent of space, [136] becomes $-P(t)\cdot u(t)$; this coupling simulates a time-dependent velocity of the walls. The induced momentum density is, to first order in u

$$m\langle j(rt)\rangle_u = m\langle j(r)\rangle_{u=0} + im^2\int_{-\infty}^{t} dr'dt'\langle [j(rt), j(r't')]\rangle\cdot u(r't'). \quad \ldots[137]$$

Suppose that

$$u(r't') = e^{ik\cdot r' - i\omega t'}e^{\eta t'}u \qquad \ldots[138]$$

where η is a positive infinitesimal small number whose effect is slowly to turn on $u(r't')$ [note that $e^{\eta t} \to 0$ as $t \to -\infty$]. Then expressing the $[j_i, j_j]$ commutator in terms of its Fourier transform $\Upsilon_{ij}(k\omega')$ and doing the ω' integral we have

$$m\langle j_i(rt)\rangle_u = m\langle j_i(r)\rangle_u - e^{ik\cdot r - i\omega t}e^{\eta t}\chi_{ij}(k, \omega + i\eta)u_j, \qquad \ldots[139]$$

where

$$\chi_{ij}(k, \omega+i\eta) \equiv \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\Upsilon_{ij}(k\omega')}{\omega+i\eta-\omega'}; \qquad \ldots[140]$$

a summation over j is understood. Our previous result [99] for $\langle g_z(r)\rangle$ is thus seen to be a special case of [139], in which $\omega+i\eta \to 0$, followed by $k \to 0$. In general, the $\omega \to 0$ and then $k \to 0$ limit of the response to a perturbation simulates having a time-independent, spatially uniform perturbation in the density matrix.

One should also note that normal mass density ρ_n, involving Υ^T, is the response to an $\omega \to 0$ and then $k \to 0$ *transverse* u, while the response to such a u that is longitudinal is the total mass density ρ. Essentially, perturbations proportional to the transverse current do not excite the superfluid while those proportional to the longitudinal current do.

We turn now to a description of the electromagnetic response of a system. The question we ask is: given a weak externally applied vector potential $A(rt)$, what is the total vector potential $A_{tot}(rt)$? The point is that the external potential induces currents in the system which themselves generate electromagnetic fields. If we define the induced vector potential by

$$A_{tot}(rt) = A(rt)+A_{ind}(rt), \qquad \ldots[141]$$

then from Maxwell's equations

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)A_{ind}(rt) = \frac{4\pi}{c}\langle J(rt)\rangle \qquad \ldots[142]$$

where $\langle J(rt)\rangle$ is the expectation value of the electromagnetic current of the system. [By taking $\langle J\rangle$ we are calculating the *average* induced vector potential.] The problem then is to calculate $\langle J(rt)\rangle$ in terms of $A(r't')$. If one neglects the field fluctuations, the coupling of the system to A_{tot} is given by [127], with A_{tot} in place of A. [Replacing the electromagnetic potential by its space and time dependent average leads to results which are accurate to lowest order in $(v/c)^2$, where v is a typical particle velocity in the system.] The induced $\langle J(rt)\rangle$ is then given, to first order in A_{tot}, by [135], with A replaced by A_{tot}.

Let us assume that $A(rt)$ is of the form

$$A(rt) = e^{ik\cdot r - i(\omega+i\eta)t}A(k\omega) \qquad \ldots[143]$$

where $A(k\omega)$ is transverse, i.e. $k\cdot A(k\omega) = 0$. Then both $\langle J(rt)\rangle$ and $A_{tot}(rt)$ will have the same space and time dependence as $A(rt)$ and will be transverse. From [135] we have

$$\langle J(rt)\rangle = -\frac{e^2}{m^2c}[nm+\chi^T(k, \omega+i\eta)]\, A^{tot}(rt), \qquad \ldots[144]$$

where

$$\chi^T(k, \omega+i\eta) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\frac{\Upsilon^T(k,\omega')}{\omega+i\eta-\omega'}. \qquad \ldots[145]$$

Finally, solving for $A_{ind}(k\omega)$ in terms of $A_{tot}(k\omega)$, from [144] and [142], and using [141] we find

$$A_{tot}(k\omega) = A(k\omega)\Bigg/\left[1-\frac{4\pi e^2\{nm+\chi^T(k,\omega+i\eta)\}}{m^2\{(\omega+i\eta)^2-c^2k^2\}}\right]. \qquad \ldots[146]$$

The denominator of [146] plays the role of a transverse susceptibility. One should note that this differs from Maxwell's definition of the transverse susceptibility; his doesn't have an $(\omega+i\eta)^2$ underneath the $4\pi e^2$. The reason for the difference is that Maxwell's transverse dielectric function relates B and H; while

$$B(rt) = \nabla\times A_{tot}(rt), \qquad \ldots[147]$$

it is not true, except in the static $(\omega+i\eta \to 0)$ limit, that $H = \nabla\times A$.

We consider now the case of a weak static, very long wavelength external magnetic field applied to the system; this corresponds to a time-independent A. Then from [146], we have

$$A_{tot}(k) = A(k)\Bigg/\left[1+\frac{4\pi e^2\{nm+\chi^T(k,0)\}}{m^2c^2k^2}\right]. \qquad \ldots[148]$$

Let us write, as for a Bose superfluid, that

$$-\lim_{k\to 0}\chi^T(k,0) = \lim_{k\to 0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon^T(k,\omega)}{\omega} \equiv \rho_n \qquad \ldots[149]$$

If ρ_n is less than $\rho = mn$, as it is in a superconductor, then the system will have a Meissner effect. This is easy to see; as $k \to 0$ [148] becomes

$$A_{tot}(k) = \frac{k^2}{k^2+(1/\Lambda^2)}A(k) \qquad \ldots[150]$$

where the length Λ is given by

$$\frac{1}{\Lambda^2} = \frac{4\pi ne^2}{mc^2}\frac{\rho_s}{\rho}, \qquad \ldots[151]$$

and

$$\rho_s \equiv \rho-\rho_n; \qquad \ldots[152]$$

Λ is in fact the penetration depth. Note that as $k \to 0$ the total electromagnetic potential approaches zero as k^2 times the external potential; thus weak slowly varying external magnetic fields will be excluded from the material. To see that Λ is the penetration depth, take the cross product of both sides of [150] by $\mathbf{k}$ and Fourier transform back to space. Then

$$(\nabla^2 - \Lambda^{-2})\mathbf{B}(\mathbf{r}) = \nabla^2 \mathbf{H}(\mathbf{r}), \quad \ldots[153]$$

when $\mathbf{H}(\mathbf{r})$ is slowly varying in space. For a constant $\mathbf{H}(\mathbf{r})$ the solutions for $\mathbf{B}(\mathbf{r})$ fall off exponentially with a characteristic length Λ inside the system; the detailed solutions of [153] depend on the boundary conditions.[7]

Thus the criterion for a Meissner effect, $\rho_n < \rho$, is exactly the same as we found for a neutral system to have a reduced moment of inertia. It is a fundamental characteristic of superfluids that ρ_n, defined from the transverse current correlation function, is less than ρ. In all systems

$$\lim_{k \to 0} \chi^L(k, 0) \to -nm,$$

as a consequence of the longitudinal sum rule[8]; at $T = 0$ in a pure system $\rho_n = 0$. The origin of the formal similarity between the Meissner effect and the reduction of the momenta of inertia is the following: a uniform magnetic field $\mathscr{H}$ can be described by a vector potential $\mathbf{A}(\mathbf{r}) = -\mathbf{r}\times\mathscr{H}/2$. The coupling [127] is to first order in $\mathscr{H}$

$$H' = -\frac{e\mathscr{H}}{2mc}\cdot \mathbf{L}$$

which is the same as a "$-\boldsymbol{\omega}\cdot\mathbf{L}$" term for a rotating system.

We should also note that [144] reduces to the London equation

$$\langle \mathbf{J}(\mathbf{r}t)\rangle = -\frac{\rho_s}{\rho}\frac{ne^2}{mc}\mathbf{A}_{\text{tot}}(\mathbf{r}t) \quad \ldots[154]$$

in the limit of fields that vary very slowly in space and time. As London showed this equation leads to zero electrical resistance since

$$\frac{\partial\langle \mathbf{J}(\mathbf{r}t)\rangle}{\partial t} = \frac{\rho_s}{\rho}\frac{ne^2}{m}\mathbf{E}(\mathbf{r}t), \quad \ldots[155]$$

where $\mathbf{E} = -c^{-1}\partial \mathbf{A}/\partial t$ is the electric field. Thus $\langle \mathbf{J}\rangle$ is constant in time in the absence of electric fields.

In a normal system

$$\lim_{k\to 0} \chi^T(k, 0) = -nm + bk^2 \quad \ldots[156]$$

where b is a positive constant. Then [148] becomes

$$\mathbf{A}_{\text{tot}}(\mathbf{k}) = \frac{\mathbf{A}(\mathbf{k})}{1+4\pi be^2/m^2c^2}; \quad \ldots[157]$$

this reduction of $\mathbf{A}_{\text{tot}}$ from $\mathbf{A}$ is just the Landau diamagnetism of normal electrons.

6

Why is $\rho_n < \rho$ in a Superfluid?

To answer this question we begin by noting that

$$\Upsilon_{ij}(-\mathbf{k}, -\omega) = \Upsilon_{ji}(\mathbf{k}\omega); \quad \ldots[158]$$

the proof follows trivially from the definition [97] of Υ. Thus we can write

$$\chi_{zz}(k,0) = -\int_0^\infty \frac{d\omega}{\pi}\frac{\Upsilon_{zz}(k\omega)}{\omega} \quad \ldots[159]$$

since Υ_{zz} is an even function of $\mathbf{k}$. Next we use the detailed balancing condition to write

$$\frac{\Upsilon_{zz}(k\omega)}{1-e^{-\beta\omega}} = \frac{m^2}{V}\int d\mathbf{r}d\mathbf{r}'\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\langle j_z(\mathbf{r}t)j_z(\mathbf{r}'t')\rangle. \quad \ldots[160]$$

For simplicity we shall restrict ourselves to $T = 0$. Then

$$\langle j_z(\mathbf{r}t)j_z(\mathbf{r}'t')\rangle = \sum_n \langle 0\,|\,j_z(\mathbf{r})\,|\,n\rangle e^{-i(E_n-E_0)(t-t')}\langle n\,|\,j_z(\mathbf{r}')\,|\,0\rangle \quad \ldots[161]$$

where $|\,0\rangle$ is the ground state, E_0 the ground state energy, and the sum is over a complete set of states; the time dependence was extracted by use of [6]. Substituting [161] into [160] and defining

$$j_{-\mathbf{k}} = \int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} j_z(\mathbf{r}) \quad \ldots[162]$$

we find

$$\Upsilon_{zz}(k\omega) = \frac{2\pi m^2}{V}\sum_n |\langle n\,|\,j_{-\mathbf{k}}\,|\,0\rangle|^2\,\delta(\omega+E_0-E_n), \quad \ldots[163]$$

and

$$\chi_{zz}(k\omega) = -\frac{2m^2}{V}\sum_n \frac{|\langle n\,|\,j_{-\mathbf{k}}\,|\,0\rangle|^2}{E_n-E_0} \quad \ldots[164]$$

Now we must study the nature of the states entering this sum in the $\mathbf{k}\to 0$ limit.[9] First of all, the momentum of the states $|\,n\rangle$ for which $\langle n\,|\,j_{-\mathbf{k}}\,|\,0\rangle$ is non-zero must be $\mathbf{k}$, since $j_{-\mathbf{k}}$ removes momentum $-\mathbf{k}$ from the ground state. Next we note that $\lim_{\mathbf{k}\to 0} j_{-\mathbf{k}} = P_z/m$ and $\langle n\,|\,P_z\,|\,0\rangle = 0$ (since the ground state has total momentum zero); invoking continuity we argue that

$$\lim_{\mathbf{k}\to 0}\langle n\,|\,j_{-\mathbf{k}}\,|\,0\rangle = 0. \quad \ldots[164a]$$

Thus as $\mathbf{k}\to 0$ the numerators in the sum [164] tend to zero. The only states $|n\rangle$ contributing to the sum are states of momentum $\boldsymbol{k}$ whose energy tends to E_0 as $\boldsymbol{k} \to \boldsymbol{0}$.

In He[4] the low-lying excitations from the ground state are phonons, the long wavelength *longitudinal* oscillations of the condensate; their energy is sk, where s, = 238 m/sec, is the sound velocity. Furthermore it can be argued[9] that the matrix element $\langle n | j_{-\boldsymbol{k}} | 0\rangle$ between the ground state and a state containing two or more phonons tends to zero at least as fast as k. Thus the only $|n\rangle$ entering [*164*] in the $\boldsymbol{k} \to \boldsymbol{0}$ limit is the state $|\boldsymbol{k}\rangle$ containing one phonon of momentum $\boldsymbol{k}$; hence in this limit

$$\chi_{zz}(\boldsymbol{k}, 0) = -\frac{2m^2 |\langle \boldsymbol{k} | j_{-\boldsymbol{k}} | 0\rangle|^2}{skV}. \qquad ...[165]$$

The essence of the argument now is that the one phonon states exhaust the sum rule in the longitudinal part of χ, and contribute nothing to the transverse part of χ. The point is that the matrix element $\langle \boldsymbol{k} | j_{-\boldsymbol{k}} | 0\rangle$ must be a vector parallel to $\boldsymbol{k}$. Thus the z component, $\langle \boldsymbol{k} | j_{-\boldsymbol{k}} | 0\rangle$, must be proportional to k_z. If $\boldsymbol{k}$ is orthogonal to the z axis then $\langle \boldsymbol{k} | j_{-\boldsymbol{k}} | 0\rangle$ vanishes, and as $\boldsymbol{k} \to \boldsymbol{0}$,

$$\chi_{zz}(\boldsymbol{k}, 0) = \chi^T(\boldsymbol{k}, 0) = \rho_n = 0. \qquad ...[166]$$

ρ_n vanishes because there are no low-lying "transverse" states; a transverse current can't excite a single longitudinal phonon. The matrix elements for the transverse current to excite a state with two or more phonons or rotons vanishes too rapidly with k to contribute to ρ_n.

On the other hand if $\boldsymbol{k}$ is along the z axis,

$$\chi_{zz}(\boldsymbol{k}, 0) = \chi^L(\boldsymbol{k}, 0) = -nm, \qquad ...[167]$$

from the longitudinal sum rule. The one phonon states exhaust this sum rule, and we infer from [*165*] and [*167*] that for small k,

$$|\langle \boldsymbol{k} | j^L_{-\boldsymbol{k}} | 0\rangle| = (skVn/2m)^{\frac{1}{2}}. \qquad ...[168]$$

The transverse current $j^T_{-\boldsymbol{k}}$ has non-vanishing matrix elements between two one-phonon states. At finite temperature where there are thermal excitations present, such matrix elements lead to a non-vanishing contribution to ρ_n.

The situation in a superconductor is entirely analogous; one also has longitudinal excitations of the condensate, known as the Anderson modes, which, as in He[4], contribute to χ^L and not to χ^T at zero temperature. There are also states corresponding to the excitation of particles from the Fermi sea. However these states in a pure superconductor are separated from the ground state by an energy gap 2Δ and hence at $T = 0$ they contribute to neither χ^L or χ^T in the long wavelength limit.

In summary, the reason that ρ_n is less than ρ in a superfluid is that the

only low-lying states are longitudinal excitations of the condensate, which contribute a finite fraction to the longitudinal sum rule but do not contribute to χ^T as $k \to 0$.

7

A Relation between ρ_s, $|\Psi|^2$ and the Fluctuations of the Condensate

While both the superfluid mass density and $m|\Psi|^2$, the condensate mass density, are non-zero in a superfluid, they are not equal. I would like to derive, for a Bose superfluid, a very general relation due to Josephson,[10] between these quantities and the fluctuation spectrum of the condensate.

As we discussed earlier the momentum density accompanying a flow of superfluid is

$$\langle \boldsymbol{g}\rangle = \rho_s \boldsymbol{v}_s. \qquad ...[169]$$

This momentum density is the response to a spatial variation in the phase of the condensate wave function. If

$$\Psi(\boldsymbol{r}) = e^{iS(\boldsymbol{r})}\,\Psi_0 \qquad ...[170]$$

where $S(\boldsymbol{r})$ is infinitesimally small and *very* slowly varying in space, then [*169*] is equivalent to the statement [cf. *35*]

$$\langle \boldsymbol{g}(\boldsymbol{r})\rangle = \rho_s \nabla S(\boldsymbol{r})/m. \qquad ...[171]$$

To derive the relation between ρ_s and $|\Psi|^2$ we calculate the response of $\langle \boldsymbol{g}\rangle$ to a variation of Ψ using the linear response theory developed in Sec. 5. The fundamental maneuver is to include a perturbation in the Hamiltonian

$$H'(t') = \int d\boldsymbol{r}'\psi^\dagger(\boldsymbol{r}'t')\zeta(\boldsymbol{r}'t') \qquad ...[172]$$

where $\zeta(\boldsymbol{r}'t')$ is an infinitesimal c-number function. [ζ plays the role of a particle source.] This term is essentially a handle on Ψ since by varying ζ we produce variations in $\Psi(\boldsymbol{r}t)$, as well as variations in $\langle \boldsymbol{g}(\boldsymbol{r}t)\rangle$.

Assume that $\zeta(\boldsymbol{r}t)$ is a long wavelength perturbation that is turned on very slowly, that is,

$$\zeta(\boldsymbol{r}t) = e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}\,e^{\eta t}\zeta, \qquad ...[173]$$

where η is a positive number which eventually will go to zero. In analogy with [*139*] this perturbation produces a variation in the condensate wave function given, for $\eta \to 0$, by

$$\delta\Psi(\boldsymbol{r}t) = \delta\langle\psi(\boldsymbol{r}t)\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\frac{A(\boldsymbol{k}\omega')}{\omega + i\eta - \omega'}\,\zeta, \qquad ...[174]$$

where the function

$$A(k\omega) = \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \langle[\psi(\mathbf{r}t), \psi^\dagger(\mathbf{r}'t')]\rangle \qquad ...[175]$$

is the spectral weight of the one-particle Green's function.[1] A describes the response as in [174] of the condensate to an external driving term, and hence it tells one the spectrum of condensate oscillations. Letting $\omega + i\eta \to 0$ we have

$$\delta\Psi(\mathbf{r}) = -e^{i\mathbf{k}\cdot\mathbf{r}} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(k\omega')}{\omega'} \zeta. \qquad ...[176]$$

The momentum density induced by ζ is similarly

$$\delta\langle \mathbf{g}(\mathbf{r})\rangle = -e^{i\mathbf{k}\cdot\mathbf{r}} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\langle[\mathbf{g}, \psi^\dagger]\rangle(k\omega')}{\omega'} \zeta \qquad ...[177]$$

as $\omega + i\eta \to 0$. The function in the integrand is given by

$$\langle[\mathbf{g}, \psi^\dagger]\rangle(k\omega) = \frac{\mathbf{k}m\omega}{k^2} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \langle[\rho(\mathbf{r}t), \psi^\dagger(\mathbf{r}'t')]\rangle. \qquad ...[178]$$

Next we substitute this expression into [177] and do the frequency integration, which produces a $\delta(t-t')$. But from [1] we have

$$\begin{aligned}\langle[\rho(\mathbf{r}), \psi^\dagger(\mathbf{r}')]\rangle &= \langle\psi^\dagger(\mathbf{r})\rangle\delta(\mathbf{r}-\mathbf{r}') \\ &= \Psi_0^*\delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \qquad ...[179]$$

Thus [177] reduces to the quite simple result

$$\delta\langle\mathbf{g}(\mathbf{r})\rangle = -e^{i\mathbf{k}\cdot\mathbf{r}} (\mathbf{k}m/k^2)\Psi_0^*\zeta. \qquad ...[180]$$

[One can equivalently derive this result from considerations of gauge invariance.]

Let us write the variation of $\Psi(\mathbf{r})$ as

$$\Psi(\mathbf{r}) = \Psi_0 + \delta\Psi(\mathbf{r}) = e^{iS(\mathbf{r})}\, \Psi_0; \qquad ...[181]$$

by a judicious choice of the phase of ζ we can arrange for $S(\mathbf{r})$ to be real in the long wavelength limit. Then combining [176] and [180], and writing

$$\mathbf{k}\delta\Psi(\mathbf{r}) = -i\nabla\delta\Psi(\mathbf{r}) = \nabla S(\mathbf{r})\Psi_0 \qquad ...[182]$$

we find

$$\delta\langle\mathbf{g}(\mathbf{r})\rangle = \frac{m\,|\Psi_0|^2}{k^2} \nabla S(\mathbf{r}) \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{A(k\omega)}{\omega}\right]^{-1}. \qquad ...[183]$$

In the long wavelength limit this equation must reduce to [171]. Thus comparing the coefficients of $\nabla S(\mathbf{r})$ we see that as $k \to 0$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{A(k\omega)}{\omega} = \frac{m^2\,|\Psi|^2}{k^2\rho_s}; \qquad ...[184]$$

this equation is the desired relation between the condensate density, the spectrum of condensate fluctuations and ρ_s.

We can use this relation, which may be regarded as a sum rule on $A(k\omega)$, to deduce some simple results on the long wavelength fluctuations of the condensate. First we note that as a consequence of the equal-time commutation relation [1], A obeys the sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(k\omega) = 1. \qquad ...[185]$$

The values of ω for which $A(k\omega)$ is non-zero are the possible changes of energy of the system when one adds a particle of momentum $\mathbf{k}$ or removes a particle of momentum $-\mathbf{k}$. What happens when one adds a particle of momentum $\mathbf{k}$ is that it drops into the condensate, increasing the energy of the system by μ, the chemical potential; but to conserve momentum the particle, when dropping into the condensate, must either create an excitation of momentum $\mathbf{k}$ (further increasing the total energy by the energy of excitation) or else absorb an excitation of momentum $-\mathbf{k}$ that is already present. At small $\mathbf{k}$ and very low temperatures, the only possible excitations are phonons, whose energy is sk. Thus the possible energy changes of the system produced by adding a particle are $\mu \pm sk$. Measuring energies with respect to μ we then expect A to vanish unless $\omega = \pm sk$, so that[11]

$$A(k\omega) = \alpha_k\delta(\omega - sk) + \alpha_k'\delta(\omega + sk). \qquad ...[186]$$

[The process of removing a particle of momentum $-\mathbf{k}$ leads to two terms of the same form.]

The two sum rules [184] and [185] enable us to determine the coefficients in [186]; the result is

$$\alpha_k = \frac{m^2\,|\Psi|^2 s}{2\rho_s k} + \tfrac{1}{2} = 1 - \alpha_k'. \qquad ...[187]$$

As $k \to 0$ the $\frac{1}{2}$ may be neglected and we find

$$A(k\omega) = \frac{\pi m^2\,|\Psi|^2 s}{\rho_s k}[\delta(\omega - sk) - \delta(\omega + sk)]. \qquad ...[188]$$

This result can be used to deduce the number of non-condensate particles $N_\mathbf{k} = \langle a_\mathbf{k}^\dagger a_\mathbf{k}\rangle$ for small k. The basic relation is that

$$N_\mathbf{k} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{A(k\omega)}{e^{\beta\omega} - 1}; \qquad ...[189]$$

this may be derived by using the detailed balancing condition, as in [90], remembering to measure energy with respect to μ. Then [188] implies that for small k

$$N_k = \frac{m^2 |\Psi|^2 s}{\rho_s k}\left[\frac{1}{e^{\beta sk}-1}+\tfrac{1}{2}\right]. \qquad \ldots[190]$$

At $T = 0$ where $\rho_s = \rho$ we find

$$N_k = \frac{ms}{2nk}|\Psi|^2 = \frac{ms}{2k}\frac{n_0}{n}, \qquad \ldots[191]$$

while for finite T ($KT \gg sk$) we have

$$N_k = \frac{m^2|\Psi|^2 KT}{\rho_s k^2}. \qquad \ldots[192]$$

Eq. [190] implies immediately that there can be no condensation in two dimensions at finite temperature,[12] or in one dimension ever. The density of non-condensate particles is given by

$$n' = n - |\Psi|^2 = \int \frac{dk}{(2\pi)^3} N_k. \qquad \ldots[193]$$

We see from [192] and [191] that this integral converges at the origin in three dimensions, but diverges in two dimensions at finite T, and always diverges in one dimension. The only way that n' can be finite (it must always be $\leq n$) in the divergent cases is for Ψ to be zero; that is, there is no condensation.

The results [190-192] are based on the form [186] for A, to which there may exist corrections at finite temperature. However it is easy to show that [192] provides an exact lower bound on N_k, in the long wavelength limit, and so the proof that $\Psi = 0$ in one or two dimensions at finite temperature becomes rigorous. The first step is to notice that the integrand $A(k\omega)/\omega$ of [184] is always ≥ 0. Furthermore, for all ω,

$$\frac{\beta\omega}{2}\coth\frac{\beta\omega}{2} = \beta\omega\left[\frac{1}{e^{\beta\omega}-1}+\tfrac{1}{2}\right] \geq 1; \qquad \ldots[194]$$

thus

$$A(k\omega)\left[\frac{1}{e^{\beta\omega}-1}+\tfrac{1}{2}\right] \geq \frac{A(k\omega)}{\beta\omega}. \qquad \ldots[195]$$

Integrating over all ω and using [185] and [189] we see that for all k

$$\int^{\infty} \frac{d\omega}{} \frac{A(k\omega)}{} \qquad \ldots[196]$$

Then from the long wavelength sum rule [184] for A, we find the desired long wavelength inequality:

$$N_k \geq \frac{m^2|\Psi|^2 KT}{\rho_s k^2} - \tfrac{1}{2}; \qquad \ldots[197]$$

as $k \to 0$ the $\frac{1}{2}$ may be neglected.

The sum rule [184] is valid only as $k \to 0$. For general k we can show however that A must obey the inequality:

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{A(k\omega)}{\omega} \geq \frac{m^2|\Psi|^2}{k^2\rho}. \qquad \ldots[198]$$

This is a special case of the Bogoliubov inequality,[13] which is based on the observation that if C is any operator, then [cf. 90]

$$\frac{\Upsilon_{C,C^\dagger}(\omega)}{1-e^{-\beta\omega}} \geq 0,$$

and hence

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{C,C^\dagger}(\omega)}{\omega} \geq 0. \qquad \ldots[199]$$

Now letting

$$C = D - B\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{B,D^\dagger}(\omega)^*}{\omega}\Big/\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{\Upsilon_{B,B^\dagger}(\omega')}{\omega'}, \qquad \ldots[200]$$

in [199], where D and B are arbitrary operators, we find

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{D,D^\dagger}(\omega)}{\omega}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{\Upsilon_{B,B^\dagger}(\omega')}{\omega'} \geq \left|\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{B,D^\dagger}(\omega)}{\omega}\right|^2; \qquad \ldots[201]$$

this is a form of the Bogoliubov inequality. To derive [198] we let $D = a_k$, the particle annihilation operator, and let

$$B = \mathbf{k}\cdot\int d\mathbf{r}\,\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\,\mathbf{g}(\mathbf{r}).$$

Then

$$\Upsilon_{D,D^\dagger}(\omega) = A(k\omega);$$

from the f-sum rule [116]

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{B,B^\dagger}(\omega)}{\omega} = nmk^2;$$

and from [177] and [180]:

$$\int^{\infty}\frac{d\omega}{2\pi}\frac{\Upsilon_{B,D^\dagger}(\omega)}{\omega} = \Psi_0^*.$$

The inequality [201] then reduces to [198]. The inequality [198] combined with [196] leads to the result that for all k

$$N_k+\tfrac{1}{2} \geqslant \frac{m^2 \,|\Psi|^2 KT}{k^2 \rho}, \qquad \ldots[202]$$

which in itself is sufficiently strong to prove the lack of condensation in one and two dimensions at finite temperatures.

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4

TRANSPORT THEORY

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1

Phenomenological Description

In this chapter we shall discuss the problem of treating non-equilibrium electronic states of a metal. The kind of question we would like to answer is: What are the electric currents and heat currents which flow when the system is driven out of equilibrium by some external agency such as an electric field or a temperature gradient? We shall limit ourselves almost entirely to the situation which is the one that is most often encountered in practice (not in optical phenomena, however) namely external disturbances which vary "slowly" in space and time and which are "weak". By a "slow" variation we mean that the external disturbances change by a very small amount over distances of atomic dimensions and over times comparable to all characteristic times of the system. The "weakness" of the external disturbances means that we shall limit ourselves to studying the response of the system to terms linear in appropriately chosen measures of the disturbances (for example, the electric field, or temperature gradients).

In addition, we shall for purposes of general discussion neglect the contributions of the positively charged ions, which form a background to the electronic motion. For metals, except at very low temperatures, these contributions are usually negligible because the ionic velocities are much less than the electronic velocities; charge and energy transport are due almost entirely to electronic motion. It is not difficult to include the ionic motions[1] (and so deal with a system of several components), but we shall not do this here.

The phenomenological equations are suggested by the following argument. If the system were in equilibrium no currents would flow.[2] As is well known,[3] in equilibrium the temperature T and chemical potential μ must be uniform throughout the system. If the variation of the driving forces is slow in space and time, then we may imagine that the system acquires a "local" equilibrium, which may be characterized by a "local" T and μ which are slowly varying functions of space and time