## 5.4 Quantization of Circulation

The velocity $\mathbf{v}_s$ possesses a fundamental property, which is characteristic of superfluid systems: its circulation along any closed curve $\Gamma$,

$$C = \int_\Gamma \mathbf{v}_s \cdot d\mathbf{l} \tag{5.46}$$

is *quantized*: it is equal to an integer multiple of a fundamental quantum $h/m$. This important property is the counterpart for the Bose liquid of the quantization of magnetic flux in superconductors.

The physical origin of the quantization is best understood by considering a one-dimensional system, of length $L$, which is closed on itself to form a ring. Since the wave-function must be unaffected by a translation from $x$ to $(x+L)$, any wave vector $q$ can only take discrete values, of the form

$$q = \frac{2\pi}{L}n \tag{5.47}$$

where $n$ is an integer. In order to set up a superfluid current, we must impart to the condensate a non-zero wave vector $q$. The current carried by the condensate is then equal to

$$J_s = \frac{N\hbar q}{m} \tag{5.48}$$

It follows from (5.47), that $J_s$ is quantized,

$$J_s = n\frac{Nh}{mL} \tag{5.49}$$

Because of the factor $N$ in (5.49), the basic quantum of current is a *macroscopic* rather than a microscopic quantity. Such a macroscopic effect of quantization is a direct consequence of the macroscopic occupation of a single quantum state. If we change the momentum of the latter by one quantum $2\pi/L$, the corresponding change in current is $N$ times larger. In a normal system, we can change the momentum of a *single* particle by an amount $2\pi/L$: instead of (5.49), the quantum of current is then $h/mL$. Since the latter quantum is vanishingly small, the quantization of current may be ignored on any macroscopic scale.

Let us return to the case of a Bose liquid. The velocity $\mathbf{v}_s$ is equal to $J_s/N$. The circulation of the velocity around the ring is simply $v_s L$, and is thus of the form

$$C = n\frac{h}{m} \tag{5.50}$$



where $n$ is an integer. We have thus proved, in this particular case, that the circulation of the superfluid velocity $\mathbf{v}_s$ is *quantized* in units $h/m$.

We now proceed to demonstrate the quantization of circulation by a more general method, one which is not restricted to a one-dimensional system. Let us replace $\mathbf{v}_s$ by its expression (5.26); the circulation $C$ along a closed curve, $\Gamma$, may then be written as

$$C = \oint_\Gamma \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m}\oint_\Gamma \operatorname{grad} S \cdot d\mathbf{l} \tag{5.51}$$

$mC/\hbar$ is thus equal to the change of $S$ when one goes around the curve (which means that $S$ is a multi-valued function). However, the wave-function of the condensate, equal to $e^{iS(\mathbf{r})}$, must be *single valued*. Thus, $S$ can only change by a multiple of $2\pi$ when one goes around a closed circuit, from which fact (5.50) follows directly. The quantization of circulation is seen to be a direct consequence of the requirement that the condensate wave-function be single-valued. It thus applies to any distribution of superfluid current.

The quantization of circulation is a major physical feature of superfluid flow. Together with the irrotational character of the flow, equation (5.44), it governs the way in which a superfluid can be set in motion. We shall illustrate in Chapter 8 the way in which these concepts operate, by considering a specific example, that of the vortical motion of a superfluid Bose liquid.

## 5.5 Flow Without Resistance: Landau Criterion

According to our previous discussion, the wave-function describing a uniform translation of the fluid is obtained by rigidly shifting the ground state in momentum space by an amount $m\mathbf{v}_s$. We now inquire to what extent such a wave-function corresponds to a *metastable equilibrium* of the liquid, displaying the characteristic feature of "resistance free flow." More explicitly, we ask whether and how the pipe walls could slow down superfluid flow. We shall thus encounter the important concept of a *critical velocity*, above which superfluid flow becomes unstable.

Let us consider a simple experiment, in which a Bose liquid flows at a constant velocity through a thin capillary tube. The corresponding superfluid velocity $\mathbf{v}_s$ must be constant throughout the cross-section of the tube. The superfluid motion is thus a *uniform* translation, in contrast with ordinary viscous flow, for which the fluid velocity varies from the walls to the center of the tube.

It is clear that any interaction of the fluid with the pipe walls cannot change the momentum of the condensed phase, as this would involve a simultaneous transition of the whole liquid, which is highly improbable. Hence, the only way by which the walls can slow down the flow is by creating elementary excitations, thereby absorbing momentum and energy from the uniform motion of the liquid. We are thus led to study the extent to which the walls can create such excitations.

We may safely assume that the walls are infinitely heavy. Therefore, in the frame of reference in which they are fixed, they transfer momentum, but not energy to the system (a point which we have earlier discussed in connection with the rotating bucket experiment). It is clear that the walls cannot create quasi-particles as long as the latter have a *positive* energy in the pipe frame of reference (since at $T = 0$ there is nothing to provide the required energy). Under such conditions, we expect superfluid flow to be stable—at least against quasi-particle creation. If, on the other hand, there exist quasi-particle states whose energy in the pipe frame is negative, multiple excitation processes become possible; they act to damp superfluid motion.

Thus far we only know the excitation spectrum of the Bose liquid in the frame of reference in which the condensed phase is at rest (with zero relative momentum). In that "condensate" frame (which has a velocity $\mathbf{v}_s$), the spectrum is just that found in the preceding section. The spectrum in the pipe frame may be obtained by means of a simple Galilean transformation. Let $\mathbf{p}_i$ be the momentum of a fluid particle, of mass $m$, as measured in the condensate frame. In the pipe frame, the system Hamiltonian is:

$$H = \sum_i \frac{(\mathbf{p}_i + m\mathbf{v}_s)^2}{2m} + \frac{1}{2}\sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$
$$= \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j) + \mathbf{p}\cdot\mathbf{v}_s + \frac{Nmv_s^2}{2} \qquad (5.52)$$

where $\mathbf{p} = \sum_i \mathbf{p}_i$ is the net momentum of the fluid particles, as measured in the condensate frame. If, first, we consider the fluid to be in its ground state in the condensate frame (with energy $E_o$, momentum $\mathbf{p} = 0$) the energy of the system as measured in the pipe frame is

$$E_o + \frac{Nmv_s^2}{2} \qquad (5.53)$$



In the same way, if we assume there is present a single quasi-particle, momentum $\mathbf{p}$, energy $\varepsilon_p$, as measured in the condensate frame, the fixed observer, tied to the pipe, will measure a system energy which is

$$E_o + \varepsilon_p + \mathbf{v}_s\cdot\mathbf{p} + \frac{Nmv_s^2}{2} \qquad (5.54)$$

It follows that in the pipe frame the quasi-particle has an energy

$$\varepsilon_p + \mathbf{p}\cdot\mathbf{v}_s \qquad (5.55)$$

Thus the walls cannot create quasi-particles as long as

$$\varepsilon_p + \mathbf{p}\cdot\mathbf{v}_s > 0 \text{ for all values of } \mathbf{p} \qquad (5.56)$$

If the condition (5.56) is met, we expect superfluid flow to be *stable.* Such a criterion was first formulated by Landau in his early work on liquid helium.

Let us suppose that the condition (5.56) fails to be satisfied for some particular values of $\mathbf{p}$ and $\mathbf{v}_s$. In this case, the walls keep creating new excitations of wave vector $\mathbf{p}$. As a result, the particular quasi-particle mode will grow exponentially in time, until that growth is limited by non-linear effects. The fluid flow is *unstable*, in that there is a steady transfer of energy and momentum from the coherent, directed motion of the condensate to an essentially incoherent group of quasi-particle modes. The instability corresponds to a transformation of the directed kinetic energy into heat, a phenomenon which is typical of viscous damping of fluid flow.

Such an instability, characterized by the sudden onset of viscosity, will occur when the liquid velocity exceeds a critical velocity, $v_c$, which is given by

$$v_c = \text{lower limit of } \frac{\varepsilon_p}{p} \qquad (5.57)$$

For $v_s < v_c$, there is no mechanism by which the fluid flow can transform its kinetic energy into heat: the flow is *superfluid* in character, being characterized by a complete absence of any viscosity.

According to the Landau criterion, (5.57), a free Bose gas should not be superfluid. In that case, $\varepsilon_p$ is equal to $p^2/2m$: for an arbitrarily low velocity $\mathbf{v}_s$, one can always find a small enough value of $\mathbf{p}$ such that (5.56) is violated. Such a conclusion is at first sight surprising, since we have seen in Chapter 4 that a free Bose gas did *not* respond to a macroscopic transverse probe; in that respect, it behaves like a genuine superfluid. Thus, there appears to be a contradiction between the two criteria for superfluid behavior, (4.10) and (3.5). The answer to this paradox is simply that the ground state of the free Bose gas is *superfluid* [as shown by (4.10)], while it is *unstable* against any motion of the fluid, however slow. This instability arises as a consequence of the parabolic nature of the excitation spectrum, and may be traced back to the absence of compressibility in the free Bose gas. The latter thus represents a very "pathological" case, which actually is very sensitive to boundary conditions (it may be shown that when the criterion (5.56) is expressed in terms of the true eigenstates of the liquid in the capillary tube, it is satisfied for low enough velocities $\mathbf{v}_s$). Such difficulties do not arise in the real case of an interacting Bose liquid: however weak the interaction, it will always give rise to a finite compressibility and sound velocity. The slope of the quasi-particle spectrum near the origin is then finite, so that the critical velocity $v_c$, defined by (5.57), no longer vanishes.

We carried out the previous analysis in a fixed frame of reference, tied to the walls of the system. It is not uninteresting to consider instead the problem from the vantage point of the moving fluid. The walls then appear as massive obstacles which move at velocity $-\mathbf{v}_s$ relative to the fluid. The persistence of superfluid flow then depends on the ability of these moving obstacles to scatter against the liquid.

Instead of a wall, let us consider a massive obstacle of microscopic size, such that its scattering against the fluid can be treated within the Born approximation. Such an obstacle behaves as a test particle which is coupled to the density fluctuations in the liquid. According to the general discussion given in Chapter 2, Vol. I, the probability per unit time that the object transfers momentum $\mathbf{p}$ and energy

$$\frac{(M\mathbf{v}_s+\mathbf{p})^2}{2M}-\frac{Mv_s^2}{2}=\mathbf{p}\cdot\mathbf{v}_s \qquad (5.58)$$

to the liquid is proportional to the dynamic form factor

$$S(\mathbf{p},\mathbf{p}\cdot\mathbf{v}_s) \qquad (5.59)$$

[see (I.2.11)]. If spontaneous creation of density fluctuation excitations in the moving fluid is to be forbidden, then the dynamic form factor for the fluid must satisfy the condition:

$$S(\mathbf{p},\mathbf{p}\cdot\mathbf{v}_s)=0 \qquad (5.60)$$

Equation (5.60) represents an obvious generalization of the Landau criterion, (5.56). It governs not only that part of the density fluctuation excitation spectrum which corresponds to the creation of a single quasi-particle excitation, but also the creation of multi-particle excitations. An advantage of (5.60) is that it provides an exact form of the stability criterion even if the single quasi-particle excitations are damped, since (5.60) governs as well the states into which the excitation decays.



Actually, the Landau criterion (5.56) is not a sufficient condition to observe superfluidity. It clearly represents a necessary condition, which prevents the spontaneous excitation of quasi-particles by the moving walls. However, it does not preclude the existence of other excitations, of a lower energy, which would be excited at lower superfluid velocities. We shall see in Chapter 8 that such excitations do in fact exist. They involve a vortex-like motion of the superfluid, leading to a sort of superfluid *turbulence*. Such a turbulence appears at velocities $v_s$ much smaller than that given by (5.57), and thus controls the stability of superfluid flow.

The Landau criterion, (5.56), is nevertheless very important, as it clearly displays the physical origin of superfluid behavior, namely the scarcity of low lying excited states (itself a consequence of Bose condensation). Furthermore, the critical velocity (5.57) marks the onset of *viscosity* in a superfluid Bose liquid (the vortex motion mentioned above corresponds to a turbulent non-viscous flow). The properties discussed in the present chapter are thus essential to our understanding of superfluidity.

## 5.6 Condensate Response as Superfluid Motion

In the course of this chapter, we have mentioned several times the intimate relationship between long wave length phonons and macroscopic superfluid motion. Such phonons involve a *longitudinal* superfluid flow, associated with density fluctuations [see (5.18)]. In order to show this connection more clearly, we consider the response to a weak scalar potential (i.e., to a test charge probe). The probe is assumed to be periodic in space and time, with wave vector $\mathbf{q}$ and frequency $\omega$. The perturbing Hamiltonian may thus be written as

$$H_{\text{ext}}=\alpha\left[\rho_{\mathbf{q}}^{+}e^{-i\omega t}+\rho_{\mathbf{q}}e^{i\omega t}\right] \qquad (5.61)$$

(corresponding to an applied potential $\alpha\cos(\mathbf{q}\cdot\mathbf{r}-\omega t)$). The perturbation (5.61) is assumed to act on the ground state $\psi_0$ of the Bose liquid.

# CHAPTER 7

# FIRST, SECOND, AND QUASI-PARTICLE SOUND

## 7.1 Collisionless vs. Hydrodynamic Regimes

We now consider the nature of the excited states of a Bose liquid at finite temperatures. In Chapters 2 and 3, we have discussed the quasi-particle excitations at finite temperatures. We here wish to concentrate on the nature of the excited states in the *long wave-length limit*. As in the previous chapters, we shall restrict our attention to states for which the condensate is very nearly uniform.

We recall that at $T = 0$, there exists a unique class of excited states in the long wave-length limit: one finds phonons at velocity $s$, the macroscopic sound velocity. Such wave propagation might be appropriately called *quasi-particle sound*, since in the long wave-length limit the only density fluctuations of importance are those produced by exciting a single quasi-particle from the condensate. Quasi-particle sound has essentially the same physical origin as zero sound in a Fermi liquid; the restoring force on a given particle comes from the averaged field of all the other particles. (We shall see how this comes about in a microscopic theory in Chapter 9).

At finite temperatures matters are not quite so simple. Quasi-particle sound is still a possible mode of excitation of the Bose liquid; however whether it is found depends on the temperature and the wave length under investigation. As one might expect, the criterion for observing quasi-particle sound is

$$\omega_{\mathbf{q}} \tau_{\mathbf{q}} \gg 1$$

where $\tau_{\mathbf{q}}$ is the life time of the quasi-particle excitation (of energy $\omega_{\mathbf{q}}$) in question. Where this criterion is satisfied, one is effectively in a *collisionless* regime in which the nature of the excitations is little changed from that at $T = 0$. The quasi-particle sound velocity will be temperature dependent, since the energy of a given quasi-particle depends on the thermal excitation of other quasi-particles.

Under most circumstances, the above criterion will not be satisfied in the extreme long wave-length limit: as $q \to 0$, $\varepsilon_q \to 0$, while $\tau_q$ will usually remain finite. Hence in the immediate vicinity of $\mathbf{q} = 0$ one is in a *hydrodynamic* regime, in which the restoring force responsible for wave propagation consists in the frequent collisions between thermal excitations which act to bring about local thermodynamic equilibrium . As was the case for the Fermi liquid, one is in the hydrodynamic limit when

$$\omega\tau_r \ll 1$$

where $\tau_r$ is the relaxation time required for achieving local thermodynamic equilibrium. It is clear that to the extent that quasi-particle collisions provide the mechanism for maintaining such equilibrium, $\tau_r$ will be the same order of magnitude as $\tau$.

We shall be interested in the density fluctuation spectrum in these two limits. We have seen in Sec. 2.6, Vol. I, that the spectrum may be specified by $\chi''(\mathbf{q},\omega)$, the imaginary part of the density-density response function. In the collisionless regime, we may expect two distinct contributions to $\chi''(\mathbf{q},\omega)$. One is a quasi-particle sound peak, which arises from excitation and de-excitation of *single* quasi-particles from the condensate. (We consider only wave lengths sufficiently long that multi-particle excitations may be neglected.) The other contribution arises from the scattering of the already thermally-excited quasi-particles. It consists in a continuous spectrum, extending from the origin to a maximum value, $qv_p$, where $v_p$ is the maximum thermal quasi-particle group velocity; it resembles the continuous spectrum found for a normal system.

The above separation is directly analogous to that carried out in the previous chapter for the current-current response functions. It may be thought of as a generalization of the two-fluid concept to frequency and wave-vector dependent quantities. We may write:

$$\chi''(\mathbf{q},\omega) = \chi''_s(\mathbf{q},\omega) + \chi''_n(\mathbf{q},\omega) \qquad (7.1)$$

with $\chi''_s(\mathbf{q},\omega)$, the superfluid component, denoting that part of $\chi''$ arising from excitation of quasi-particles from the condensate. In the collisionless regime, the two components of $\chi''$ will be distinct, the extent of their overlap being proportional to the (quite-small) probability that a thermal quasi-particle of momentum $\mathbf{q}$ decay into two quasi-particles of momentum $\mathbf{q} - \mathbf{p}$ and $\mathbf{p}$ as a result of scattering against the condensate. We shall show that these two components satisfy separate sum rules. Unlike the case at $T = 0$, sum rule considerations do not enable us to pin down precisely the quasi-particle sound velocity; they do, however, yield an order-of-magnitude estimate of the extent of its departure from the macroscopic, zero-temperature, value $s$.



In the hydrodynamic regime, one finds two modes of wave propagation, first and second sound. Superfluid hydrodynamics is richer than its Fermi liquid counterpart because one can have relative motion of the thermal quasi-particles and the condensate. In the first sound mode, the normal fluid and superfluid components move in phase with each other; it will be seen to be primarily a density wave, which resembles closely the hydrodynamic sound mode of a normal liquid. Second sound, on the other hand, corresponds to an out-of-phase motion of the two components; it is a wave motion characterized primarily by a periodic variation in the temperature of the system. One expects to see two peaks in $\chi''(\mathbf{q},\omega)$ at the first and second sound frequencies; however, the amplitude of the second sound peak will be quite small, since second sound involves only a very slight fluctuation in the particle density. We shall see that sum rules permit us to fix the relative amplitudes of these two peaks.

We study first the propagation of first and second sound in the hydrodynamic limit and then go on to a consideration of quasi-particle sound in the collisionless regime. As usual, the transition from one regime to the other takes place for frequencies $\omega$ and relaxation times $\tau$ such that $\omega\tau \approx 1$. It corresponds here to a transition from first sound to quasi-particle sound, and will be characterized by a maximum in sound-wave attenuation. We discuss the experimental evidence for such a transition at the close of the chapter.

Our considerations will be confined to *reversible* phenomena. There are, of course, a wide variety of irreversible phenomena which have as their physical origin collisions between the thermally-excited quasi-particles. A most successful phenomenological account of such collisions, and their consequences, has been developed by Landau and Khalatnikov (1949). We refer the interested reader to Khalatnikov's book [Khalatnikov (1965)] for an account of their theory.

## 7.2 Two-Fluid Hydrodynamics: First and Second Sound

In order to derive the modes of propagation of first and second sound it is necessary that one consider the generalization of the two-fluid model to non-equilibrium situations, as set forth by Tisza and Landau. The microscopic basis for such a model resembles closely that used for the thermal quasi-equilibrium situation. The normal fluid consists in the thermal quasi-particles; if these are in equilibrium in a frame of reference moving at a velocity $\mathbf{v}$ with respect to the condensate, they are described by a distribution function

$$n_p(\mathbf{v}) = \frac{1}{e^{\beta(\epsilon_p - \mathbf{p}\cdot\mathbf{v})} - 1} \tag{7.2}$$

This concept may be generalized to describe a non-equilibrium situation provided it is such that $\mathbf{v}$ and $T$ are slowly-varying functions of space and time. Such variation should be sufficiently slow that one can still speak of *local* thermodynamic equilibrium, a condition which requires that spatial variations be slow compared to a mean free path, and temporal variations slow compared to the relaxation time needed to establish such local equilibrium. One considers as well variations in space and time of both the density and velocity of the condensate.

We shall here confine our attention to reversible fluid motion; we neglect the viscous effects which, as we have mentioned, necessarily accompany motion of the thermally-excited quasi-particles. The four basic equations which describe fluid flow then take a simple form: two conservation laws and two dynamic equations. The first conservation law is that of the local density of the system:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \tag{7.3}$$

while the second is conservation of entropy. Let the entropy per unit volume of the system be $\tilde{S}$; since it is carried solely by the thermally-excited quasi-particles, the conservation law reads:

$$\frac{\partial \tilde{S}}{\partial t} = -\nabla \cdot \left(\tilde{S}\mathbf{v}_n\right) \tag{7.4}$$

The first dynamic equation governs condensate motion and is that we have derived in Chapter 5,

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla\left(\mu + \frac{1}{2}\rho \mathbf{v}_s^2\right) \tag{7.5}$$

The second dynamic equation governs motion of the entire liquid and is

$$\frac{d\mathbf{J}}{dt} = -\nabla P \tag{7.6}$$



where $P$ is the local pressure acting on the liquid.

The three "new" equations, (7.3), (7.4), and (7.6) are nearly obvious; one can scarcely imagine them taking any other form. They may be derived by writing down the Boltzmann equation for the quasi-particle distribution function, and taking advantage of the conservation laws to eliminate the collision term. We refer the interested reader to, for example, the lectures of de Boer (1963), for the details of such a derivation.

It is illuminating to write the two dynamic equations in slightly different form; this may easily be done with the aid of the thermodynamic identity:

$$\rho d\mu = -\tilde{S}dT + dP \tag{7.7}$$

On making use of (7.7), and further, keeping only linear terms in the equations of motion, we find, on combining (7.5) and (7.6),

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} = -\frac{\rho_s}{\rho}\nabla P + \frac{\rho_s}{\rho}\tilde{S}\nabla T \tag{7.8}$$

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_n}{\rho}\nabla P - \frac{\rho_s}{\rho}\tilde{S}\nabla T \tag{7.9}$$

We see that a pressure gradient acts to drive both fluids in the same direction, while a temperature gradient acts to drive them in opposite directions. This latter aspect may be exhibited explicitly if one multiplies (7.5) by $\rho$ and subtracts the resulting equation from (7.6): one finds, on using (7.7),

$$\rho_n \frac{\partial}{\partial t}(\mathbf{v}_n - \mathbf{v}_s) = -\tilde{S}\nabla T \tag{7.10}$$

A temperature gradient acts as an "osmotic" pressure, which tends to drive the fluids in opposite directions.

The above set of four basic equations may be reduced to two simple differential equations which govern the liquid motion. The first of these, the density equation of motion, is found by taking the time derivative of (7.3) and substituting on the right-hand side the appropriate result from (7.6); on keeping only linear terms, one finds thereby

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 P \tag{7.11}$$

The second, the entropy equation of motion, is obtained in similar fashion from (7.4), (7.9); making use of (7.11), we may write it as

$$\frac{\partial^2 \tilde{S}}{\partial t^2} = \frac{\tilde{S}^2}{\rho}\frac{\rho_s}{\rho_n}\nabla^2 T + \frac{\tilde{S}}{\rho}\frac{\partial^2 \rho}{\partial t^2} \tag{7.12}$$

This equation takes a yet simpler form, if one introduces the entropy per unit mass,

$$S = \frac{\mathcal{S}}{\rho}\,; \tag{7.13}$$

one then finds

$$\frac{\partial^2 S}{\partial t^2} = S^2 \frac{\rho_s}{\rho_n} \nabla^2 T \tag{7.14}$$

Periodic solutions for (7.11) and (7.14) are obtained by considering small departures of the pressure and temperature from equilibrium, according to

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \delta S \tag{7.15a}$$

$$\delta T = \left(\frac{\partial T}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial T}{\partial S}\right)_\rho \delta S \tag{7.15b}$$

One searches, furthermore, for periodic solutions of the coupled equations, of the form:

$$\rho = \rho_o + \delta\rho \exp\left[i\left(\mathbf{q}\cdot\mathbf{r} - \omega t\right)\right] \tag{7.16a}$$

$$S = S_o + \delta S \exp\left[i\left(\mathbf{q}\cdot\mathbf{r} - \omega t\right)\right] \tag{7.16b}$$

The resulting density and entropy wave propagation is readily found if one neglects thermal expansion of the liquid, for then a change of pressure is not accompanied by a change in temperature, and, vice versa, a change in temperature is not accompanied by a change in density. Such an approximation is equivalent to assuming that $C_p = C_v$, and is quite accurate at the very low temperatures we consider. On making it, one sees at once that temperature and density waves are decoupled.

Indeed, on substituting equations (7.16) and (7.15) into (7.11) and (7.14), and continuing to keep only linear terms, we find the following dispersion relations for wave propagation:

*Density Waves (First Sound)*

$$\omega^2 = s_1^2 q^2\,; \quad s_1^2 = \left(\frac{\partial P}{\partial \rho}\right)_S \tag{7.17}$$

*Entropy Waves (Second Sound)*

$$\omega^2 = s_2^2 q^2\,; \quad s_2^2 = \frac{\rho_s}{\rho_n} S^2 \left(\frac{\partial T}{\partial S}\right)_\rho = \frac{\rho_s}{\rho_n}\left(\frac{S^2 T}{C_v}\right) \tag{7.18}$$



The velocity of the density wave is essentially that of the usual first sound wave in a normal liquid; the second sound velocity is seen to depend intimately on the presence of a superfluid component, and vanishes near the $\lambda$-point.

On referring back to (7.10), we see that in a first sound wave, the normal and superfluid components move in phase with each other: $\mathbf{v}_n = \mathbf{v}_s$. In a second sound wave, they move out of phase, in such a way that the net matter transport is negligible; according to (7.6),

$$\mathbf{J} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0 \tag{7.19}$$

in a second sound wave.

It is not overly difficult to take into account the small coupling between density and temperature waves. One simply makes use of the complete equations, (7.15), and finds:

$$\left(\frac{\omega^2}{s_1^2 q^2} - 1\right)\delta\rho + \left(\frac{\partial P}{\partial S}\right)_\rho \left(\frac{\partial \rho}{\partial P}\right)_S \delta S = 0 \tag{7.20}$$

$$\left(\frac{\partial T}{\partial \rho}\right)_S \left(\frac{\partial S}{\partial T}\right)_\rho \delta\rho + \left(\frac{\omega^2}{s_2^2 q^2} - 1\right)\delta S = 0 \tag{7.21}$$

The condition that the equations be compatible yields a quadratic equation for $\omega^2$, viz:

$$\left[\frac{\omega^2}{s_1^2 q^2} - 1\right]\left[\frac{\omega^2}{s_2^2 q^2} - 1\right] = \left[1 - \frac{C_v}{C_p}\right] \tag{7.22}$$

on making use of the appropriate thermodynamic identities. The right-hand side of (7.22) is, in fact, very small. It vanishes at $T = 0$, and is only $7 \times 10^{-4}$ at $T = 1.5°$K [London (1954)]. As a result, the two modes of wave propagation are effectively uncoupled, and their velocities accurately specified by (7.17) and (7.18).

The precise extent to which, for example, second sound involves a density fluctuation may be determined by simple sum rule considerations applied to the density fluctuation excitation spectrum. The relevant sum rules are those derived in Sec. 2.6, Vol. I, which for convenience, we reproduce here:

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega\, \chi''(\mathbf{q},\omega)\,\omega = \frac{Nq^2}{m} \tag{7.23}$$

$$\lim_{\mathbf{q}\to 0}\left\{-\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega\, \frac{\chi''(\mathbf{q},\omega)}{\omega}\right\} = \frac{N}{m s_i^2} \tag{7.24}$$

where $s_i$ is the *isothermal* sound velocity. Let us assume that $\chi''$ contains both a first and second sound peak, according to

$$\chi''(\mathbf{q},\omega) = -\frac{\pi N}{2m}\left\{\frac{Z_1}{s_1}\left[\delta(\omega - s_1 q) - \delta(\omega + s_1 q)\right] + \frac{Z_2}{s_2}\left[\delta(\omega - s_2 q) - \delta(\omega + s_2 q)\right]\right\} \tag{7.25}$$

This expression is somewhat oversimplified, since both peaks will be broadened by viscous damping effects; however, as long as the peaks do not overlap (a situation well-satisfied in practice), the form, (7.25), suffices for considerations based on sum rules. On substituting (7.25) into the sum rules, and making use of the thermodynamic relation,

$$s_1^2 = (C_p/C_v)\, s_i^2 \tag{7.26}$$

one finds readily

$$Z_2 = \frac{C_p/C_v - 1}{s_1^2/s_2^2 - 1} \ll 1 \tag{7.27}$$

$$Z_1 = 1 - Z_2 \cong 1 \tag{7.28}$$

for the strength of the two poles in the density fluctuation spectral density. The result, (7.27), is in full accord with our general conclusion, based on (7.22), that the admixture of density fluctuation in a temperature wave will be of order $(C_p/C_v - 1)$.

Second sound is one of the most spectacular manifestations of the superfluid behavior of He II. Its propagation has been studied by many different experimental techniques. In Fig. 7.1 we plot the theoretical variation of $s_2$ with temperature. The experimental results obtained are in excellent agreement with the theoretical curve down to temperatures of the order of 0.7°K. Below this temperature, they begin to depart, for a very simple reason. At such temperatures the mean free path of a phonon is of the order of the size of the experimental apparatus, so that one no longer has sufficiently frequent collisions between the thermal excitations to establish the local thermodynamic equilibrium. (At $T = 0.8$°K, the calculations of Landau and Khalatnikov show that the phonon mean free path is of the order of 0.1 mm; it is greater than 1 cm at 0.5°K.) Under such circumstances, if one introduces a heat pulse at one end of the system (thereby creating phonons) a given excitation may simply propagate to the other end (at the quasi-particle sound velocity) without suffering any collisions. Experimentally one observes substantial distortion of

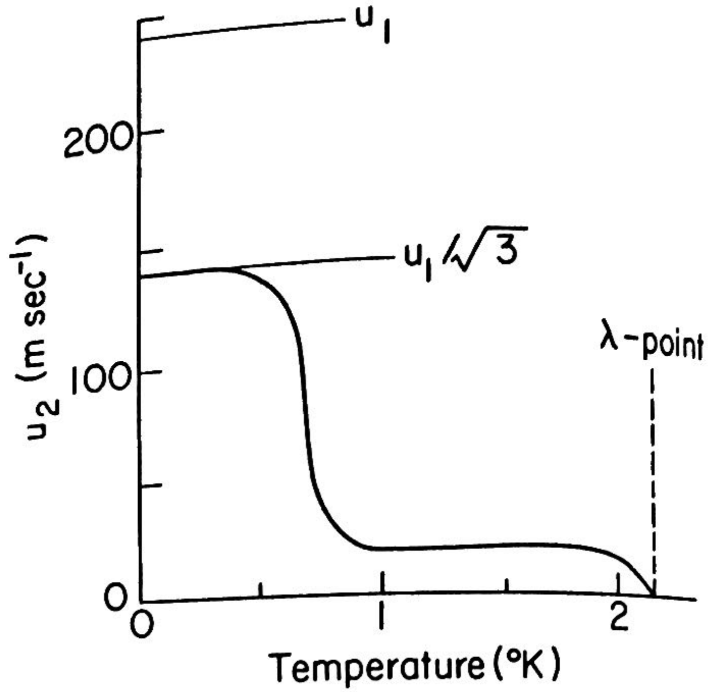


**FIGURE 7.1.** *The velocity of second sound at the vapor pressure [after Atkins (1959)].*

a temperature pulse of a sort which is consistent with such a physical picture.

We note from Fig. 7.1 that in the very low temperature region ($T \lesssim$ 0.5°K), the theoretical value of the second sound velocity approaches a constant; one finds in fact

$$s_2 \simeq \frac{s_1}{\sqrt{3}} \quad (T \le 0.5°\mathrm{K}) \tag{7.29}$$

This result may readily be found from (7.18); at such temperatures the only thermally-excited quasi-particles which are of importance are the phonons, whose velocity will be seen to be nearly $s$ in this temperature region. It is straightforward to calculate $\rho_n$, $S$, and $C_v$ for this phonon "gas"; it is left as an exercise for the reader to show that one thereby obtains (7.29).

Indeed, in this rather inaccessible region (from an experimental point of view, in view of the long phonon mean free paths), there exists an alternative derivation of the second sound mode. It is based on the fact that at such temperatures second sound corresponds to a change in the density of the phonons (associated with the change in temperature). What and Wilks (1951, 1952) have shown that under those circumstances second sound may be viewed as a compression wave in an "almost ideal" phonon gas. The relation, (7.29), is then comparable to that which exists for compression waves in an ideal gas of molecules, where one finds the familiar relation, $c = \bar{u}_1/\sqrt{3}$, between the velocity of the compression wave and the average speed of a molecule. From this point of view it is obvious that the mode cannot exist once the phonon mean free path becomes comparable to the wave-length of the mode.

## 7.3 Quasi-Particle Sound

We consider now the behavior of the density fluctuation spectrum in the collisionless regime. We shall be interested in wave lengths $\lambda$ such that

$$\xi \ll \lambda \ll \ell \tag{7.30}$$

where $\xi$ is a coherence length (of the order of the interparticle spacing) and $\ell$ is the mean free path for the quasi-particle excitations under study. Where (7.30) is satisfied, one is both in the long wave-length regime and the collisionless regime. Under these circumstances, there will be two distinct contributions to $\chi''(\mathbf{q},\omega)$. We have identified these in (7.1) as a superfluid part, $\chi''_s(\mathbf{q},\omega)$, arising from excitation and de-excitation of single quasi-particles from the condensate, and a normal component, $\chi''_n(\mathbf{q},\omega)$, produced by the scattering of already thermally-excited quasi-particles.

The condition, (7.30), is not overly difficult to satisfy in practice. It is met, for example, in the neutron scattering experiments at $T = 1°\text{K}$ for the lower range of $q$ values studied ($0.2\ \text{Å}^{-1} \le q \lesssim 0.6\ \text{Å}^{-1}$). Let us consider briefly the finite temperature analysis of such an experiment, which measures directly the dynamic form factor, $S(\mathbf{q},\omega)$.

As for $\chi''(\mathbf{q},\omega)$, there will be distinct contributions to $S(\mathbf{q},\omega)$, arising from transitions involving the condensate, and from scattering of the thermal quasi-particles. Since the latter contribution extends over a continuous range of frequencies, it is not easily separated from the background in an experiment; in practice, one measures only the "superfluid" component, $S_s(\mathbf{q},\omega)$, associated with the scattering of single quasi-particles in or out of the condensate. Inspection of (I.2.162) shows that we may write:

$$S_s(\mathbf{q},\omega) = NZ_q\left\{\delta(\omega-\omega_q) + n_q\left[\delta(\omega-\omega_q) + \delta(\omega+\omega_q)\right]\right\} \tag{7.31}$$



In (7.31) the first term in brackets corresponds to quasi-particle excitation; the second and third represent the induced excitation and de-excitation of single quasi-particles, respectively. The result, (7.31), is the natural finite-temperature analogue of the single quasi-particle part of $S(\mathbf{q},\omega)$, (2.19). Both the quasi-particle energy, $\omega_q$, and the transition probability, $Z_q$, depend on temperature, since the interaction of a given quasi-particle with the thermally-excited quasi-particles will be different from that it has with the condensate. We shall see that at 1°K, any departures of $\omega_q$ and $Z_q$ from their zero-temperature values are too small to be picked up in a neutron scattering experiment in the collisionless regime. It is for this reason that we have taken the measurements of Henshaw, Woods, et al. at 1°K as a direct measure of the quasi-particle spectrum at $T = 0$. Let us emphasize that at wave vectors less than about $0.6\ \text{Å}^{-1}$, they have observed quasi-particle sound.

We note, too, that for temperatures $T$ and wave-vectors $q$ such that

$$\beta\omega_q \gg 1 \tag{7.32}$$

thermally-induced excitation and de-excitation of single quasi-particles is negligible. This condition is likewise met in the experiment of Henshaw and Woods at 1.1°K. We may thus justify, a posteriori, the use of a "zero-temperature" dynamic form factor, (2.19), in the analysis of their neutron scattering experiments.

We turn now to a consideration of the temperature dependence of the quasi-particle phonon energy, $\omega_q$, or what is equivalent, the quasi-particle sound velocity, $s(T)$. We first establish two new sum rules for $\chi''(\mathbf{q},\omega)$ in the collisionless regime; these permit us to obtain a qualitative measure of such temperature dependence. We then compare experimental measurements of $s(T)$ with theoretical calculations of this quantity in the very low temperature regime in which the only thermal excitations present are phonons.

To derive the sum rules we note first that current conservation provides the following relation between $\chi''_{\parallel}(\mathbf{q},\omega)$, the imaginary part of the longitudinal current-current response function, and $\chi''(\mathbf{q},\omega)$:

$$\chi''_{\parallel}(\mathbf{q},\omega) = \frac{\omega^2}{q^2}\chi''(\mathbf{q},\omega) \tag{7.33}$$

By analogy with (7.1), we can therefore separate $\chi''(\mathbf{q},\omega)$ into a superfluid and a normal component, associated with condensate transitions and thermal quasi-particle transitions respectively. The components $\chi_{\parallel}^{n}$ and $\chi_{\parallel}^{s}$ obey the separate Kramers-Kronig relations:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega \frac{\chi_{\parallel}^{n''}(\mathbf{q},\omega)}{\omega} = \chi_{\parallel}^{n}(\mathbf{q},o) \tag{7.34a}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega \frac{\chi_{\parallel}^{s''}(\mathbf{q},\omega)}{\omega} = \chi_{\parallel}^{s}(\mathbf{q},o) \tag{7.34b}$$

Moreover, we have seen in the preceding chapter that in the long wave-length limit

$$\lim_{\mathbf{q}\to 0} \chi_{\parallel}^{n}(\mathbf{q},o) = \chi_{\perp}(\mathbf{q},o) = -\frac{\rho_n}{m^2} \tag{7.35a}$$

$$\lim_{\mathbf{q}\to 0} \chi_{\parallel}^{s}(\mathbf{q},o) = -\frac{\rho_s}{m^2} \tag{7.35b}$$

On making use of (7.33) and (7.35), we see that equation (7.34) provides directly the following sum rules:

$$\lim_{\mathbf{q}\to 0}\left[-\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega \chi_n''(\mathbf{q},\omega)\omega\right] = \frac{\rho_n q^2}{m^2} \tag{7.36}$$

$$\lim_{\mathbf{q}\to 0}\left[-\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega \chi_s''(\mathbf{q},\omega)\omega\right] = \frac{\rho_s q^2}{m^2} \tag{7.37}$$

The two sum rules represent an effective split-up of the $f$-sum rule, (7.23), in the long wave-length collisionless regime, into separate sum rules for $\chi_n''$ and $\chi_s''$. We have therefore only one *new* sum rule in this regime. It is natural to inquire for what values of $q$ the various "long wave-length" sum rules, (7.24), (7.36), and (7.37) are valid. The condition for their validity is that there exist a *local* relation between the external field and the responding quantity. Thus the compressibility sum rule is valid as long as there exists a local relation between the external force field and the induced density; the superfluid sum rules are valid when one has a local relation between the induced current and the external "vector potential." Such local relations apply provided the wave-lengths of interest are long compared to the coherence lengths which characterize the system under study.

We now consider the information which the long wave-length sum rules provide on the quasi-particle sound velocity. The superfluid component, $\chi_s''(\mathbf{q},\omega)$, may be directly obtained from (7.31). We have

$$\chi_s''(\mathbf{q},\omega) = -\pi\left[S_s(\mathbf{q},\omega) - S_s(\mathbf{q},-\omega)\right]$$



$$= -\pi N Z_q\left[\delta(\omega-\omega_q) - \delta(\omega+\omega_q)\right] \tag{7.38}$$

On substituting (7.38) into (7.37), we find

$$\omega_q Z_q = \left(\frac{\rho_s}{\rho}\right)\left(\frac{q^2}{2m}\right) \tag{7.39}$$

A second relation between $\omega_q$ and $Z_q$ is provided by the compressibility sum rule, (7.24). On substituting (7.38) and (7.1) into (7.24), we find:

$$\frac{Z_q}{\omega_q} = \frac{1}{2ms_i^2} + \lim_{\mathbf{q}\to 0}\frac{1}{2\pi N}\int_{-\infty}^{+\infty} d\omega \frac{\chi_n''(\mathbf{q},\omega)}{\omega} \tag{7.40}$$

The two relations, (7.39), and (7.40) are not sufficient to determine $\omega_q$, since we do not know $\chi_n''(\mathbf{q},\omega)$. We may, however, obtain a rough estimate of the integral in (7.40) by noting that the transitions contributing to $\chi_n''$ involve the scattering of a thermal quasi-particle from state $\mathbf{p}$ to state $(\mathbf{p}+\mathbf{q})$. The corresponding excitation energy is equal to $\mathbf{q}\cdot\mathbf{v}_p$. We may thus write

$$\frac{\int_o^\infty \chi_n''(\mathbf{q},\omega)\omega d\omega}{\int_o^\infty \frac{\chi_n''(\mathbf{q},\omega)d\omega}{\omega}} = q^2\overline{v^2}$$

where $\overline{v^2}$ is of the order of the average squared group velocity of thermal quasi-particles. On making use of (7.36), we find

$$\lim_{\mathbf{q}\to 0}\left[\frac{1}{\pi}\int_o^\infty \frac{\chi_n''(\mathbf{q},\omega)}{\omega}d\omega\right] = \frac{\rho_n}{m^2\overline{v^2}} = \frac{\rho_n}{\rho}A(T)\frac{N}{ms_i^2} \tag{7.41}$$

where we have set

$$A(T) = \frac{s_i^2}{\overline{v^2}} \tag{7.42}$$

$A(T)$ is a temperature dependent constant, which is of order unity, and which is subject to the inequality,

$$\frac{\rho_n}{\rho}A(T) < 1 \tag{7.43}$$

since the left-hand side of (7.40) is positive definite. On making use of (7.41), we obtain our desired result:

$$\omega_q = s_i q\left\{\frac{1-\frac{\rho_n}{\rho}}{1-\frac{A\rho_n}{\rho}}\right\}^{1/2} \tag{7.44}$$

An approximate microscopic calculation of the temperature-dependent quasi-particle sound velocity has been given by Hohenberg and Martin (1964). The present considerations permit one to set a lower limit on $s(T)$; according to (7.43), one has

$$s(T) > \left(\frac{\rho_s}{\rho}\right)^{1/2} s_i \tag{7.45}$$

Let us inquire whether such a temperature variation can be seen in neutron scattering experiments. We note that at 1.1°K, where the neutron scattering experiments have been carried out in the long wave-length regime, $\rho_n/\rho \sim 10^{-3}$. We therefore expect a shift from the zero temperature sound velocity which is of the order of 0.1%. Such an accuracy likely cannot be achieved in neutron experiments. However, to the extent that one can remain in the collisionless, long wave-length regime at higher temperatures, one may expect to see a measurable shift in the quasi-particle sound velocity from its zero-temperature value. Indeed, for temperatures near the $\lambda$-point, this sound velocity may tend toward zero, a temperature dependence which would be strikingly different from that observed (and anticipated) for the first sound velocity [Atkins (1959)].

The very slight dependence on temperature of the quasi-particle sound velocity in the low temperature regime (0.1°K to 0.8°K) has been observed by Whitney and Chase (1962), who use direct ultrasonic pulse experiments to measure the sound velocity at a frequency of 1 Mc. Their experimental results for the shift in the sound velocity from its zero-temperature value are shown in Fig. 7.2. Using Khalatnikov's calculations for the phonon relaxation times, one finds that $\omega\tau \sim 1$ for $T \lesssim 0.8$°K, so that the decrease observed in that vicinity may be attributed to the onset of hydrodynamic behavior, while the initial *increase* clearly takes place in the collisionless regime. The magnitude of the increase is not large ($\sim$ 1 cm/sec at $T = 0.4$°K); we now see to what extent it may be explained theoretically.

We may attempt an approximate calculation of $s(T)$ by calculating $\chi_n''(\mathbf{q},\omega)$ for a non-interacting gas of quasi-particles, and then using (7.41) to obtain $A(T)$. By starting with the exact expression for $\chi''(\mathbf{q},\omega)$, (I.2.165), and following steps directly analogous to those used to derive (6.22), one finds

$$\chi_n''(\mathbf{q},\omega) = -\pi \sum_{\mathbf{q}} \left(n_{\mathbf{p}} - n_{\mathbf{p+q}}\right) F_{\mathbf{pq}}^2 \delta\left(\omega - \epsilon_{\mathbf{q+p}} + \epsilon_{\mathbf{p}}\right) \tag{7.46}$$



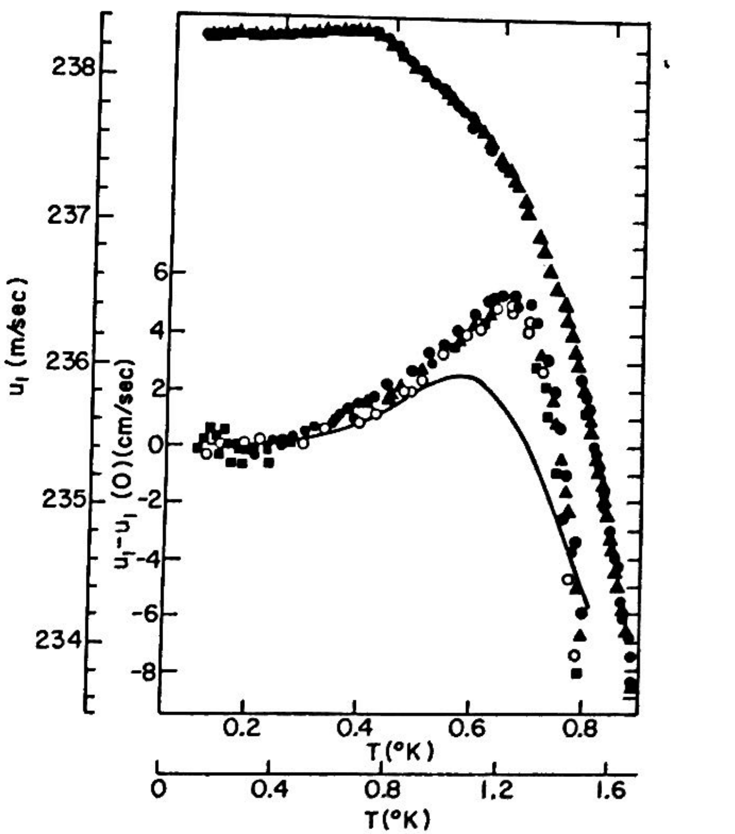


FIGURE 7.2. *Velocity of sound in liquid helium II. The lower curve shows the region of the maximum on an expanded scale [from Whitney and Chase (1962)].*

where $F_{\mathbf{pq}}$ is the matrix element for scattering of a quasi-particle from a state $\mathbf{p}$ to a state $\mathbf{p+q}$,

$$F_{\mathbf{pq}} = \langle \mathbf{p+q} | \sum_{\mathbf{p}'} C_{\mathbf{p}'+\mathbf{q}}^{+} C_{\mathbf{p}'} | \mathbf{p} \rangle \tag{7.47}$$

Current conservation provides a relation between $F_{\mathbf{pq}}$ and the corresponding matrix element for the longitudinal current density fluctuation,

$$\left(\epsilon_{\mathbf{p+q}} - \epsilon_{\mathbf{p}}\right) F_{\mathbf{pq}} = \langle \mathbf{p+q} | \sum_{\mathbf{p}'} C_{\mathbf{p}'+\mathbf{q}}^{+} C_{\mathbf{p}'} \left(\mathbf{p}' \cdot \frac{\mathbf{q}}{m}\right) | \mathbf{p} \rangle \tag{7.48}$$

We now assume that the current fluctuations are symmetric (as would be the case for non-interacting quasi-particles); the longitudinal current fluctuation matrix element should then be equal to its transverse counterpart. According to (6.21) and (6.25), the latter is $(\mathbf{p}\cdot\eta_{\mathbf{q}}/m)$. We therefore would expect, in the long wave-length limit,

$$F_{\mathbf{pq}} = \frac{\mathbf{q}\cdot\mathbf{p}}{m\mathbf{q}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}} \tag{7.49}$$

and

$$\chi_n''(\mathbf{q},\omega) - \frac{1}{\pi}\sum_{\mathbf{p}}\frac{\mathbf{q}\cdot\nabla_{\mathbf{p}}n_{\mathbf{p}}}{(\mathbf{q}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}})^2}\left(\frac{\mathbf{q}\cdot\mathbf{p}}{m}\right)^2\delta(\omega-\mathbf{q}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}})$$

$$= -\frac{1}{\pi}\sum_{\mathbf{p}}\left(\frac{\partial n}{\partial\varepsilon}\right)\left\{\frac{\left(\mathbf{q}\cdot\frac{\mathbf{p}}{m}\right)^2}{\mathbf{q}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}}\right\}\delta(\omega-\mathbf{q}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}) \tag{7.50}$$

On substituting (7.50) into (7.36), we see that the above expression is consistent with the normal quasi-particle sum rule, as it should be. We next evaluate the contribution of $\chi_n''(q,\omega)$ to the compressibility sum rule, (7.41). The calculation may be done readily in the temperature regime in which the only thermal quasi-particles of importance are the phonons, $T \leq 0.5°\text{K}$. One finds

$$A(T) = 3\left(\frac{\rho_n}{\rho}\right)\left(\frac{s_i}{s}\right)^2 \tag{7.51}$$

If we now substitute (7.51) into (7.43), and keep only the lowest order terms in $\rho_n/\rho$, we obtain

$$s(T) = s_i(T)\left\{1+\frac{\rho_n}{\rho}\right\} \tag{7.52}$$

The non-interacting excitation gas calculation of $s(T)$ thus predicts a modest increase over the isothermal sound velocity $s_i$; the origin of the increase lies in the fact that at these temperatures the thermal quasi-particle contribution to the compressibility sum rule increases more rapidly with increasing temperature than does their contribution to the $f$-sum rule. At higher temperatures this is likely not the case.

The shift (7.52) has the right sign to explain the measurements of Whitney and Chase; however, its magnitude is far too small, since $\delta s$ at 0.4°K is of the order of 1 cm/sec. Coherence effects in the scattering of thermally-excited phonons thus must play an important role. A calculation in which such effects are taken into account has been carried out by Andreev and



Khalatnikov (1963); we refer the interested reader to their paper for the details of the calculation. They find (in c.g.s. units)

$$s(T) = s(0) + 20\,T^4\ln\left(\frac{67}{T^2}\right)\ (T \lesssim 0.5°\text{K}) \tag{7.53}$$

The temperature variation implied by (7.53) is considerably more rapid than that found using (7.52). The agreement with experiment is consequently better, though not perfect, as may be seen in Fig. 7.2.

## 7.4 Transition from Quasi-Particle Sound to First Sound

We have emphasized that in the vicinity of $\omega\tau \sim 1$, one gets a transition from quasi-particle sound to first sound. If one works at fixed frequency and increases the temperature, one expects to find quasi-particle sound at the lowest temperatures, then a rather complicated transition region, followed by first sound in the "high temperature" region for which $\omega\tau \ll 1$. We have just seen that the velocity of "sound" measured by Whitney and Chase, shows a maximum at a temperature such that $\omega\tau \sim 1$. Below that temperature (~ 0.75°K), one finds quasi-particle sound with a velocity which increases with temperature: above it, the hydrodynamic sound velocity decreases with increasing temperature, in accordance with theoretical expectations.

A rather more striking manifestation of the transition region is found in ultrasonic attenuation experiments. In the very low temperature, quasi-particle sound regime, the sound wave attenuation will simply be proportional to $1/\tau$, the lifetime of a given phonon at that frequency. On the other hand, at "high" temperatures where one is in the hydrodynamic regime, the sound wave attenuation is determined by the appropriate viscosity coefficient. The corresponding attenuation is reduced over that found in the collisionless regime by a factor of $(\omega\tau)^2$, and is therefore proportional to $\tau$. As in the case of the Fermi liquid, the general behavior of the ultrasonic attenuation coefficient with temperature should therefore be governed by an expression of the form

$$\frac{A}{\tau}\left\{\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\right\} \tag{7.54}$$

According to (7.54), one expects to find a maximum in the ultrasonic attenuation coefficient in the region $\omega\tau \sim 1$. This is exactly what is observed, as may be seen from the results of Chase and Herlin at 12.1 Mc,

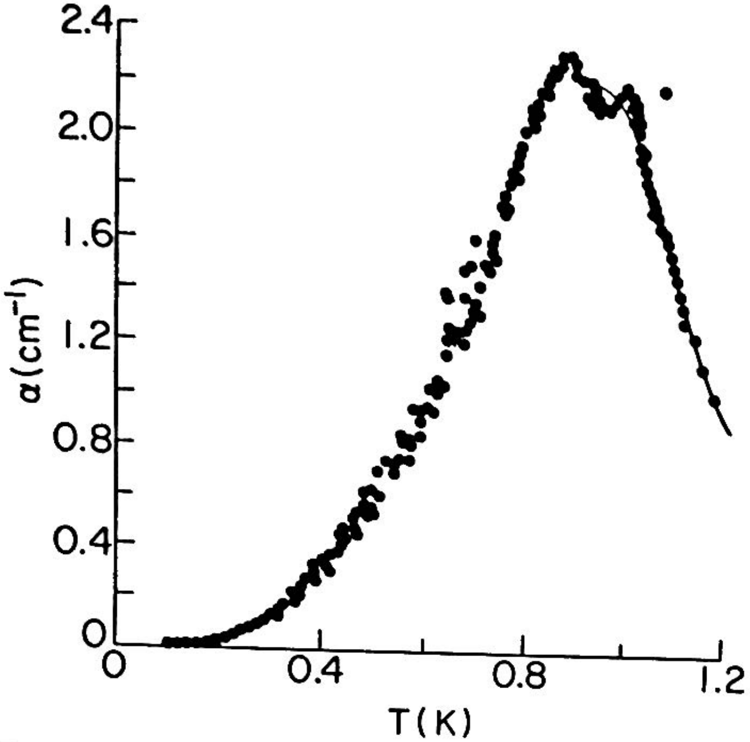


**FIGURE 7.3.** *Attenuation of sound in liquid helium at 12.1 Mc/sec [from Chase and Herlin (1955)].*

shown in Fig. 7.3. Above 0.8°K, the observed attenuation is in good agreement with the theory of Khalatnikov, whose results are shown there. Below 0.8°K, agreement between theory and experiment is not satisfactory, essentially since new physical processes come into play in the attenuation of quasi-particle sound at these lower temperatures. Let us emphasize that we lack a precise treatment of the transition regime, since it comprises just that regime for which neither first-sound nor quasi-particle sound offer an altogether satisfactory description of the system behavior.

# CHAPTER 8

# VORTEX LINES

## 8.1 Structure of a Vortex Line

In the preceding chapters, we have discussed at length the longitudinal oscillations of a superfluid Bose liquid. We now focus our attention on "steady" superfluid flow, in which the superfluid velocity, $\mathbf{v}_s$, satisfies the equation

$$div\ \mathbf{v}_s = 0$$

Such flow can, in principle, occur only in multiply-connected systems. However, it is possible to achieve an equivalent situation in the bulk of the liquid by setting up vortex lines. In a *vortex line* $\gamma$, the fluid rotates around the curve $\gamma$; the flow is irrotational everywhere *except on* $\gamma$. As a consequence, the velocity increases as one approaches $\gamma$. At a small enough distance, the centrifugal force is (in principle) large enough to overcome the capillary force, so that one expects to find a narrow cylindrical hole in the liquid along the vortex line. The existence of the hole makes the system multiply connected. Actually, such a picture is physically false, as the computed radius of the expected hole is comparable to the interparticle spacing: on such a small scale, the concept of a fluid flow is meaningless. We should therefore consider a vortex line as an irrotational motion of the fluid around the line $\gamma$, down to distances at which the hydrodynamic equations are no longer valid.

It should be realized that vortex lines are not peculiar to superfluids. They constitute as essential feature of the dynamics of normal fluids, and have, indeed, been extensively studied since the 19th century. Many of the results which we shall discuss are actually "classics" of hydrodynamics.

The major new feature brought in by superfluidity, and a most important one indeed, is the quantization of vortex lines, arising as a consequence of the quantization of circulation. In cases where quantization may be neglected, the properties of vortex lines in a superfluid are very close to those in a normal fluid.

We consider first a straight vortex line, in which the fluid rotates around the $z$-axis. The superfluid velocity at an arbitrary point $M$ is tangential, as shown in Fig. 8.1; it depends only on the distance $r$ between $M$ and the axis of the vortex. In order for the flow to be irrotational, the magnitude of the velocity must vary as

$$v_s = \frac{k}{r} \tag{8.1}$$

where $k$ is a constant. Equation (8.1) is valid for large distances. At small distances, when $r$ becomes comparable to the coherence length, $\xi$, $\mathbf{v}_s$ varies rapidly, and the hydrodynamic description breaks down. We may thus consider that the vortex possesses a core of radius $\xi$, inside which the microscopic structure of the fluid is appreciably altered. For liquid He II, this core is of the order of a few Å wide.

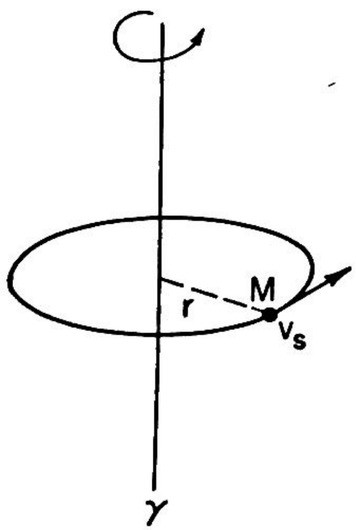


FIGURE 8.1. *Geometry of a straight vortex line.*



The circulation of the velocity (8.1) around a circle of radius $r$ is equal to $2\pi k$. On taking account of the quantization of circulation, (5.50), we may write (8.1) as

$$v_s = \frac{n\hbar}{mr} \tag{8.2}$$

where $n$ is an integer, giving the number of quanta of circulation in the vortex line (the sign of $n$ determines the direction of the flow). We shall see that at low enough temperatures, only vortex lines with $|n| = 1$ are excited.

The energy per unit length of the vortex line, $E$, may be interpreted as a "tension" of the line. The kinetic energy associated with the rotation is given by

$$\int_o^R 2\pi r dr \frac{Nm}{2} v_s^2 = \frac{\pi N}{m} n^2 \hbar^2 \int_o^R \frac{dr}{r} \tag{8.3}$$

where $R$ is the radius of the vessel containing the fluid. The integral, (8.3), diverges for small $r$; however, there exists a natural cutoff at the coherence length, $\xi$. With logarithmic accuracy, we can thus write

$$E = \frac{\pi N}{m} n^2 \hbar^2 \log \frac{R}{\xi} \tag{8.4}$$

In principle, we should add to (8.4) the change in potential energy brought about by the vortex. Actually, it may be shown that this correction affects only the immediate vicinity of the core ($r \sim \xi$); the corresponding contribution to the energy is comparable to the uncertainty inherent to the logarithmic accuracy of (8.4), and may thus be neglected.

To the extent that the finite size of the vortex core may be neglected, the velocity field $\mathbf{v}_s$ given by (8.2) satisfies the relation

$$\text{curl } \mathbf{v}_s = C\eta\delta_2(\mathbf{r}) \tag{8.5}$$

where $\eta$ is a unit vector along the vortex axis, while

$$C = \frac{nh}{m} \tag{8.6}$$

is the circulation around the vortex line. $\delta_2(\mathbf{r})$ is a two-dimensional $\delta$-function in the plane perpendicular to the vortex line. Equation (8.5) may be extended to describe the properties of curved vortices, such as vortex rings (a vortex line closed upon itself). In that more general case, $\delta_2(\mathbf{r})$ is defined at every point $M$ of the vortex line as a $\delta$-function in the plane normal to the vortex; $\eta$ is a unit vector tangent to the vortex line

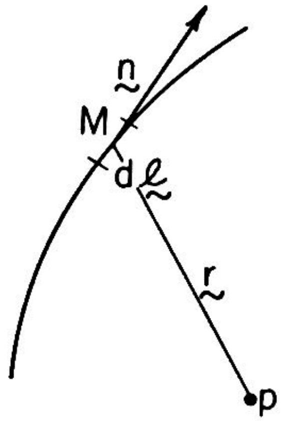


**FIGURE 8.2.** *Geometry of a curved vortex line.*

at $M$ (Fig. 8.2). The velocity $\mathbf{v}_s$ created by a curved vortex at a point $P$ cannot generally be obtained in closed form; however, it may be written as a line integral along the vortex:

$$\mathbf{v}_s = \frac{C}{4\pi}\int_\gamma \frac{\eta \times \mathbf{r}}{r^3} d\ell \tag{8.7}$$

(where $\mathbf{r}$ is the vector going from $M$ to $P$). The expression (8.7) is formally similar to that giving the magnetic field created by a current carrying wire; its proof is left as an exercise to the reader. The velocity pattern of such curved vortices is complicated; at a distance $r$ which is small compared to the radius of the curvature of $\gamma$, $\mathbf{v}_s$ may be written as

$$\mathbf{v}_s = \frac{C}{2\pi r^2}\eta \times \mathbf{r} + \bar{\mathbf{v}} \tag{8.8}$$

The first term on the right-hand side of (8.8) arises from that part of the vortex which lies in the immediate vicinity of the point $P$ under study. The other, regular, contribution $\bar{\mathbf{v}}$ describes the velocity created at point $P$ by the rest of the vortex.

The analysis is easily extended to the case of several vortices. The total velocity $\mathbf{v}_s$ is simply the sum of the velocities arising from each vortex. Let us for instance consider two parallel straight vortex lines $\gamma_1$ and $\gamma_2$, at a distance $d$ from each other, possessing respectively $n_1$ and $n_2$ quanta of circulation. The net velocity is

$$\mathbf{v}_s = \mathbf{v}_1 + \mathbf{v}_2 \ ; \tag{8.9}$$



the total kinetic energy per unit length along the vortices is

$$\frac{Nm}{2}\iint d\sigma\,(v_1 + v_2)^2 \tag{8.10}$$

The square terms in (8.10) correspond to the energy of the single vortex lines, while the cross term describes an *interaction energy* between two vortex lines. After a straightforward integration, one finds that the interaction energy is equal to

$$E_{12} = 2\pi\frac{N}{m}n_1 n_2 \hbar^2 \log\frac{R}{d} \tag{8.11}$$

where $R$ is the radius of the vessel. [(8.11) is valid only if the two vortex lines in question are not too close to the vessel boundary.]

The total energy of the two vortex lines is equal to

$$E = \frac{\pi N}{m}\hbar^2\left\{(n_1^2 + n_2^2)\log\frac{R}{\xi} + 2n_1 n_2 \log\frac{R}{d}\right\} \tag{8.12}$$

which we write in the form

$$E = \frac{\pi N}{m}\hbar^2\left\{(n_1 + n_2)^2\log\frac{R}{\xi} - 2n_1 n_2 \log\frac{d}{\xi}\right\} \tag{8.13}$$

For a given value of the *total* circulation, one thus obtains a lower energy by having two vortices, with $n_1$ and $n_2$ circulation quanta respectively, in place of a single one with $(n_1 + n_2)$ quanta. Consequently, the lowest energy will be achieved by having a large number of vortex lines, each with a single quantum ($n = 1$), and no vortices of higher order. At $T = 0$, we thus expect to observe only "elementary" vortex lines, containing a single circulation quantum.

## 8.2 Dynamics of a Vortex Line

The motion of a vortex line is governed by a very simple law: each point $M$ of the vortex moves at the velocity which the fluid possesses at $M$ itself. More exactly, in the vicinity of $M$, the velocity $\mathbf{v}_s$ may be written in the form (8.8): the point $M$ then moves at velocity $\bar{\mathbf{v}}$. Consider, for example, the two parallel vortices described earlier; each moves at the velocity created at its core by the other. If $n_1 = n_2$, the two vortices rotate around each other (Fig. 8.3a); if, instead, $n_1 = -n_2$, the vortices are subject to a uniform translation (Fig. 8.3b). Another example of such