

Relativistic kinematics all

1. Natural units
2. 4-vectors, Lorentz transformations
3. Co- and contra-variant notation, metric tensor, invariants
4. 2-body decay in CM: monocromaticity, limit cases, 3-momentum
5. 2-body decay in flight: minimum and maximum energy, energy distribution.
6. 2-body decay in flight: angles and Blaton ellipses, minimum and maximum angles

5 Natural units

Carrying around $\hbar$ & c is cumbersome.

① Use eV for energy $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$

② $\hbar = 6.58 \times 10^{-16} \text{ eV} \cdot \text{s}$

$c = 2.99792458 \times 10^{23} \text{ fm/s}$ $1 \text{ fm} = 10^{-15} \text{ m}$

$\hbar c = 197 \text{ MeV} \cdot \text{fm}$

③ Let's ~~keep~~ define the unit of length so that

$c = 1 \rightarrow [L] = [T]$

• Let's define the unit of mass so that $\hbar = 1$

$[M] = [E] = [P] = [L^{-1}] = [T^{-1}]$

④ Everything can be expressed in eV or eV^{-1}

Transformation in other units can be done

inserting proper $\hbar$ & c factors

$1 \text{ MeV}^{-1} = 197 \text{ fm}$

$1 \text{ s} = 3 \times 10^{23} \text{ fm}$

$1 \text{ MeV} = \frac{1}{197} \text{ fm}^{-1} = \frac{300 \times 10^{21}}{197} \text{ s}^{-1} = 1.52 \times 10^{21} \text{ s}^{-1}$

etc. $1 \text{ s}^{-1} = 6.5 \times 10^{-16} \text{ eV}$

Relation between lifetime and width $\Gamma, \tau = \hbar$

Coming from $\Delta E \cdot \Delta t \geq \hbar$

$\tau = \frac{\hbar}{\Gamma} = \frac{1}{\Gamma}$

1 Lorentz transformations (LT) (special relativity) GR.3

- a. Physics is the same in all inertial systems
- b. Speed of light is the same in all systems

If S' is moving along z with velocity v , then coordinates transform as

$$(1) \quad \begin{cases} ct' = \gamma(ct - \beta z) \\ x' = x \\ y' = y \\ z' = \gamma(z - \beta ct) \end{cases} \quad \begin{aligned} \beta &= v/c < 1 \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} > 1 \end{aligned}$$

Inverse transformation ($v \rightarrow -v$, $\beta \rightarrow -\beta$)

$$(2) \quad \begin{cases} ct = \gamma(ct' + \beta z') \\ x = x' \\ y = y' \\ z = \gamma(z' + \beta ct') \end{cases} \quad (\text{Ex: derive from eq (1)})$$

4-vectors $x^\mu = (x^0, x^1, x^2, x^3) = (\cancel{ct}, x, y, z) = (t, \vec{r})$
 $p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p})$

Invariants : $a \cdot b = a^0 b^0 - (\vec{a} \cdot \vec{b})$
 $a^2 = a^0^2 - |\vec{a}|^2$

Metric (flat) $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$a \cdot b = a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu$ (Covariant)

$p^\mu p_\mu = p^2 = E^2 - \vec{p}^2 = m^2 \quad p = m u = m(\gamma, \gamma \vec{\beta})$

$p^\mu p^\nu$ = 4x4 tensor, no contraction!

Vector types a^2 is invariant $a^2 < 0$ Space-like
 $a^2 = 0$ light-like

Momentum p^μ : $p^2 = m^2 \geq 0$ $a^2 > 0$ time-like
↳ time or light like

Conservation: 4-momentum is conserved in all known processes
 $\sum P_{in}^\mu = \sum P_{out}^\mu$

- Rest or Center-of-mass frame: $\sum_i \vec{p}_i' = 0$

Use a "*" to identify CM quantities

- Laboratory system: where measurements are taken.

Fixed-target experiments: the target is at rest in the LAB.

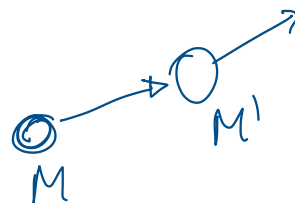
- Particle decays in CM (Δ is implied)

- ① Can M decay in M' and nothing else?

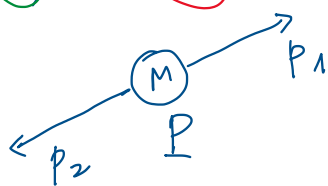
$$P_i^M = P_f^{M'} \quad (M, \vec{0}) = (E', \vec{p}')$$

$$\rightarrow \vec{p}' = 0 \rightarrow E' = M \quad M' = M$$

$$\rightarrow P_i^2 = P_f^2 \rightarrow M^2 = M'^2$$



- ② 2 body decay



$$P^M = p_1^M + p_2^M \quad [P = p_1 + p_2]$$

$$(M, \vec{0}) = (E_1, \vec{p}_1) + (E_2, \vec{p}_2)$$

A) E_1 and E_2 are fixed in the CM - Monochromatic

First try: correct but lengthy.

$$M = E_1 + E_2$$

$$\rightarrow \sqrt{m_1^2 + |\vec{p}_1|^2} + \sqrt{m_2^2 + |\vec{p}_2|^2} = M$$

$$\vec{p}_1 = -\vec{p}_2 \Rightarrow |\vec{p}_1| = |\vec{p}_2| \rightarrow$$

USE INVARIANTS!

$$(P - p_1) = p_2 \rightarrow (P - p_1)^2 = p_2^2 \rightarrow M^2 + m_1^2 - 2Pp_1 = m_2^2$$

$$2Pp_1 = 2(M, \vec{0}) \cdot (E_1, \vec{p}_1) = 2ME_1$$

$$M^2 + m_1^2 - 2ME_1 = m_2^2$$

MEMORIZE!

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

Kinetic energy $T \equiv E - M$ (useful for heavy nuclei)

$$T_1 = E_1 - m_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} - m_1 = \frac{(M - m_1)^2 - m_2^2}{2M}$$

$$T_2 = \frac{(M - m_2)^2 - m_1^2}{2M}$$

Limit cases

$$m_1 = m_2$$

$$E_1 = E_2 = \frac{M}{2}$$

$$\pi^0 \rightarrow \gamma\gamma$$

$$\rho^0 \rightarrow \pi^+ \pi^-$$

$$m_2 = 0$$

$$E_1 = \frac{M^2 + m_1^2}{2M} = \frac{M}{2} \left(1 + \frac{m_1^2}{M^2} \right)$$

$$m_1 = 0$$

$$m_2 = 0$$

$$E_2 = \frac{M}{2} \left(1 - \frac{m_1^2}{M^2} \right)$$

$$\pi^+ \rightarrow \mu^+ \nu_\mu$$

3-momentum

if E is fixed, also $|\vec{p}|$ is fixed in the CM

$$|\vec{p}_1|^2 = E_1^2 - m_1^2 = \frac{1}{4M^2} [M^4 + m_1^4 + m_2^4 + 2M^2 m_1^2 - 2M^2 m_2^2 - 2m_1^2 m_2^2 - 4M^2 m_1^2] =$$

$$= \frac{1}{4M^2} \lambda(M^2, m_1^2, m_2^2)$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

$$= [a - (\sqrt{b} + \sqrt{c})^2][a - (\sqrt{b} - \sqrt{c})^2]$$

Källén
triangle
function

Three examples with numbers

a) $^{152}_{62}\text{Sm}^* \rightarrow ^{152}_{62}\text{Sm} + \gamma$

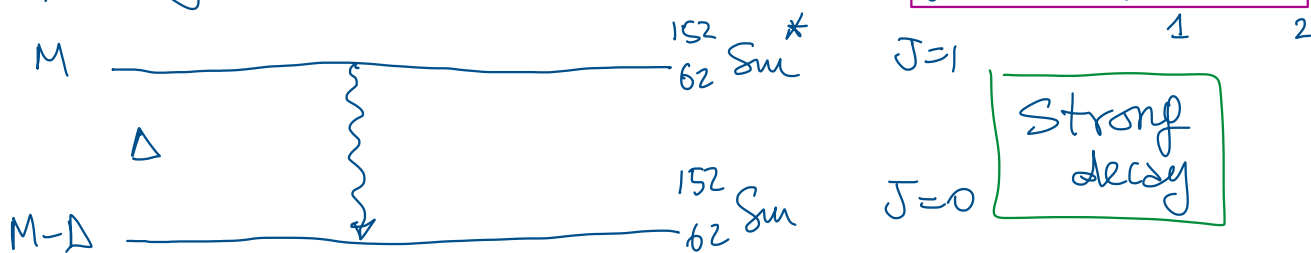
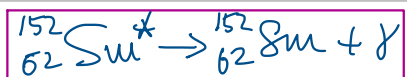
d) is $\gamma \rightarrow e^+ e^-$ possible?

b) $\pi^0 \rightarrow \gamma\gamma$

no: $(p_{e^+} + p_{e^-})^2 \geq 4m_e^2$

c) $\pi^+ \rightarrow \ell^+ \nu$ $\left\langle \begin{matrix} e^+ \nu_e \\ \mu^+ \nu_\mu \end{matrix} \right.$

a) Heavy nucleus de-excitation



$$E_\gamma = \frac{M^2 + m_\gamma^2 - (M-\Delta)^2}{2M} = \frac{2M\Delta - \Delta^2}{2M} = \Delta - \frac{\Delta^2}{2M}$$

$$E_\gamma < \Delta$$

nucleus recoil energy!

$$\text{Check: } T_1 = \frac{(M-m_1)^2 - m_2^2}{2M} = \frac{\Delta^2}{2M}$$

For $\Delta \sim 1 \text{ MeV}$
 $M_{\text{Sm}} \sim 150 \text{ GeV}$

$$\frac{\Delta^2}{2M} = \frac{(10^6)^2 \text{ eV}}{2 \times 150 \times 10^9} = \frac{10^{12}}{0.3 \times 10^{12}} \approx 33 \text{ eV}$$

b) $\pi^0 \rightarrow \gamma\gamma$

EM decay $\tau \sim 10^{-17} \text{ s}$

$$m_{\pi^0} = 135 \text{ MeV}$$

$$c\tau = 3 \times 10^8 \text{ m/s} \times 10^{-17} \text{ s} = 3 \times 10^{-9} \text{ m} = 3 \text{ nm}$$

$$\Delta l \text{ in lab} = \gamma \beta c \tau$$

$$E_\gamma = \frac{m_{\pi^0}}{2} = 67.5 \text{ MeV}$$

c) $\pi^+ \rightarrow \ell^+ \nu$

$$m_{\pi^+} = 140 \text{ MeV}$$

$$\tau^+ = 26 \text{ nsec}$$

$$\pi^+ \rightarrow \mu^+ \nu \sim 100\%$$

$$m_{e^+} = 511 \text{ keV}$$

$$c\tau = 7.8 \text{ m}$$

$$m_{\mu^+} = 105 \text{ MeV}$$

Weak decay

$$E_\ell = \frac{m_\pi^2 + m_\ell^2 - m_\nu^2}{2m_\pi} = \frac{m_\pi}{2} \left[1 + \frac{m_\ell^2}{m_\pi^2} \right]$$

$$T_\ell = \frac{(m_\pi - m_\ell)}{2m_\pi}$$

$$E_\nu = \frac{m_\pi}{2} \left[1 - \frac{m_\ell^2}{m_\pi^2} \right]$$

$$e: m_e \ll m_\pi \quad \frac{m_e}{m_\pi} = \frac{0.5}{140} = 280 \text{ O.R.}$$

$$E_e = E_\nu = T_e \approx 70 \text{ MeV} \quad \gamma = \frac{E_e}{m_e} = 140$$

$$\mu: \frac{m_\mu}{m_\pi} = \frac{105}{140}$$

$$E_\nu = |\vec{p}_\mu| = 30 \text{ MeV}$$

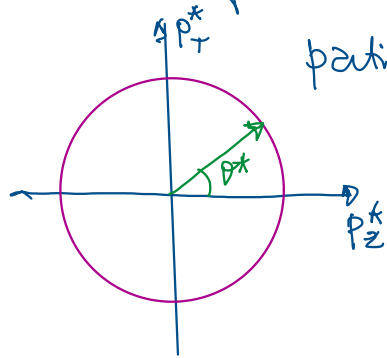
$$E_\mu \sim 110 \text{ MeV} \quad \gamma = \frac{E}{m} = 1.04$$

$$T_\mu \sim 4.1 \text{ MeV} \quad \beta = \frac{p}{E} = 0.27$$

NON RELATIV.

2-body decay in flight

Now use * for CM quantity



particle 1:

$$p_z^{*2} + p_T^{*2} = |\vec{p}_1^*|^2 \text{ fixed}$$

only variable is the angle θ^*

in the LAB: Energy (min-max)

Angle (min-max)

Lorentz transformations

$$\begin{cases} E = \gamma (E^* + \beta p_z^*) = \gamma (E^* + \beta p^* \cos \theta^*) \\ p_z = \gamma (p_z^* + \beta E^*) = \gamma (p^* \cos \theta + \beta E^*) \\ p_T = p_T^* \end{cases}$$

(the two particles have different p^* in general)

$E_{\min}, E_{\max}$

$$\rightarrow \cos \theta^* = \pm 1 \quad E_{\min} = \gamma (E^* - \beta p^*)$$

$$E_{\max} = \gamma (E^* + \beta p^*)$$

Example $\pi^0 \rightarrow \gamma\gamma$ with $E_{\pi^0} = 14 \text{ GeV}$

$$\gamma = \frac{E_{\pi^0}}{m_{\pi^0}} = \frac{14}{0.14} = \underline{100}$$

$$\text{for } \gamma \quad E^* = p^* = \frac{m_{\pi^0}}{2}$$

$$(1 - \beta) = \frac{1 - \beta^2}{1 + \beta} \approx \frac{1}{2\gamma^2} = 5 \times 10^{-5}$$

$$E_{\max} = \gamma E^* (1 + \beta) \approx 2 \gamma E^* = 2 \frac{E_{\pi^0}}{m_{\pi^0}} \frac{m_{\pi^0}}{2} = E_{\pi^0} = 14 \text{ GeV}$$

$$E_{\min} = \gamma E^* \underbrace{(1 - \beta)}_{\neq 0} = \gamma E^* \frac{1}{2\gamma^2} = \frac{m_{\pi^0}}{4\gamma} = \frac{140 \text{ MeV}}{400} = 0.35 \text{ MeV}$$

- for v.r. particles $1 + \beta = 2$ but $1 - \beta \neq 0$!
- NEVER set $\beta = 1$

Energy distribution

$\frac{dN}{dE}$ in the LAB

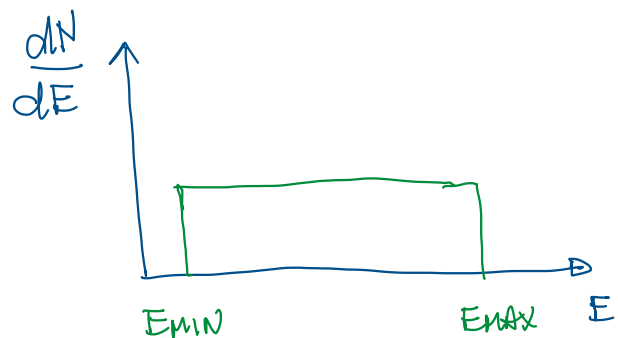
$$E = \underbrace{\gamma(E^* + \beta p^* \cos\theta^*)}_{\text{constants}} \quad \uparrow \text{only variable}$$

$$\frac{dN}{dE} = \underbrace{\frac{dN}{d\cos\theta^*}}_{\text{cm angular distribution}} \underbrace{\left(\frac{d\cos\theta^*}{dE}\right)^{-1}}_{\text{only variable}} = \frac{1}{\gamma\beta p^*}$$

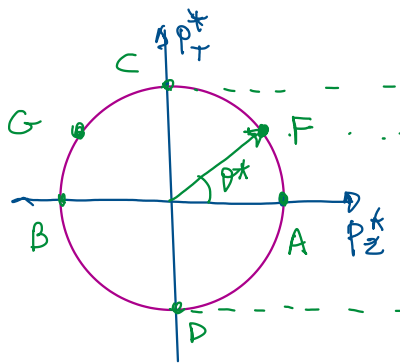
cm angular distribution

IF cm distribution is isotropic (e.g. scalar particle)

$$\Rightarrow \frac{dN}{dE} = \text{flat}$$



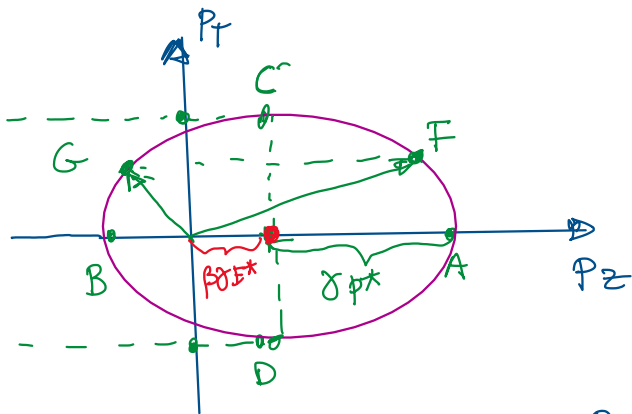
Angles



$$p_T^{*2} + p_z^{*2} = p^{*2}$$

$$\frac{p_T^{*2}}{p^{*2}} + \frac{p_z^{*2}}{p^{*2}} = 1$$

circle



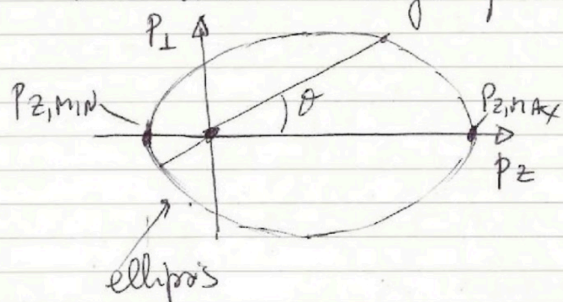
$$\begin{cases} p_z = \gamma p_z^* + \beta\gamma E^* \\ p_T = p_T^* \end{cases}$$

$$p_z^* = \frac{p_z - \beta\gamma E^*}{\gamma}$$

$$p_T^* = p_T$$

$$\frac{(p_z - \beta\gamma E^*)^2}{\gamma^2 p^{*2}} + \frac{p_T^2}{p^{*2}} = 1 \quad \text{Ellipsis}$$

a) if $\beta_{cm} < \beta^*$ the particles going backwards will continue to go backwards



$$p_z = \gamma E^* (\beta^* \cos \theta + \beta_{cm})$$

$$p_{z,max} = \gamma E^* (\beta^* + \beta_{cm}) > 0$$

$$p_{z,min} = \gamma E^* (-\beta^* + \beta_{cm}) < 0$$

$$0 < \theta < 2\pi$$

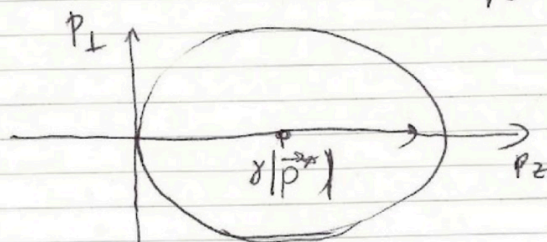
If the distribution in angle θ^* was uniform, in θ it will be forward peaked

Always this case for massless particles ($\beta^* = 1$)

$$\Delta p_z = p_{z,max} - p_{z,min} = \gamma (+|\vec{p}_c^*| + \beta_{cm} E^*) - \gamma (-|\vec{p}_c^*| + \beta_{cm} E^*)$$

$$= 2\gamma (|\vec{p}_c^*|) \rightarrow \text{expanded by } \gamma.$$

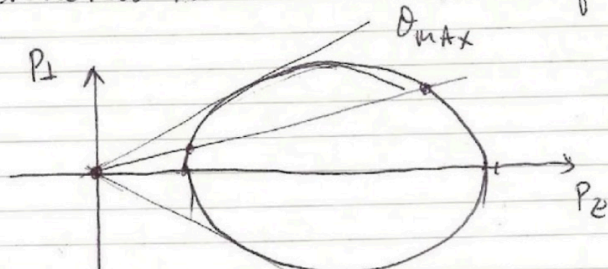
b) if $\beta_{cm} = \beta^*$ then particles going backwards will have $p_z = 0$ in the lab



$$p_{z,min} = 0$$

Center of the ellipsoids:
 $\gamma |\vec{p}^*|$

c) if $\beta_{cm} > \beta^*$ the even particles moving backwards in cm will move forward in the lab

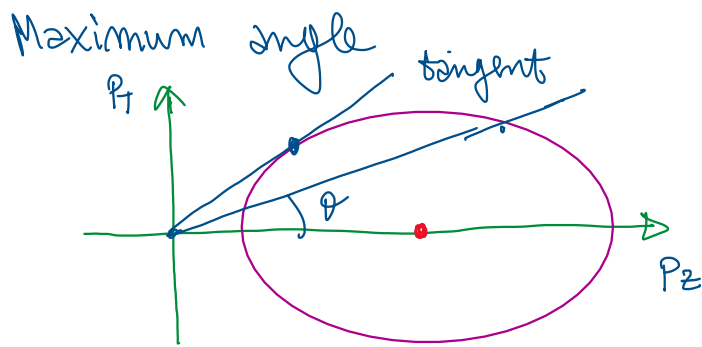


$$p_{z,min} > 0$$

• There is a maximum angle in the lab

• For a given angle in the lab there are two possible angles in the CM.

one finds $\tan^2 \theta_{max} = \frac{\beta^{*2}}{\gamma^2 (\beta_{cm}^2 - \beta^{*2})}$



$$\beta \gamma E^* > \gamma p^*$$

$$\beta > \frac{p^*}{E^*} = \beta^*$$

$$\left(\frac{p_z - \beta \gamma E^*}{\gamma^2 p^{*2}} \right)^2 + \frac{p_T^2}{p^{*2}} = 1$$

$$\left(\frac{p_z - \gamma \beta E^*}{\gamma^2 p^{*2}} \right)^2 + \frac{\tan^2 \theta p_z^2}{p^{*2}} = 1$$

$$\left\{ \begin{array}{l} p_T = \tan \theta p_z \rightarrow \text{fixed angle in LAB} \end{array} \right.$$

in general 2 solutions for p_z . Max angle when coincide.

$$(p_z - \gamma \beta E^*)^2 + \gamma^2 \tan^2 \theta p_z^2 - \gamma^2 p^{*2} = 0$$

$$p_z^2 [1 + \gamma^2 \tan^2 \theta] - 2 \gamma \beta E^* p_z + \gamma^2 (\beta^2 E^{*2} - p^{*2}) = 0$$

$$\frac{\Delta^2}{4} = (\gamma \beta E^*)^2 - \underbrace{(1 + \gamma^2 \tan^2 \theta)}_{\text{unknown}} \gamma^2 (\beta^2 E^{*2} - p^{*2}) = 0 \quad \times \frac{1}{\gamma^2 E^{*2}}$$

$$\beta^* = \frac{p^*}{E^*}$$

$$\beta^2 - (\beta^2 - \beta^{*2}) = \gamma^2 \tan^2 \theta (\beta^2 - \beta^{*2})$$

$$\tan^2 \theta = \frac{\beta^{*2}}{\gamma^2 (\beta^2 - \beta^{*2})}$$

$$\tan \theta = \frac{1}{\gamma} \frac{1}{\sqrt{(\beta/\beta^*)^2 - 1}} \quad [\beta > \beta^*]$$

-> This is the maximum angle of one particle wrt z axis, NOT the maximum opening angle between the two particles

