

$$\underbrace{\frac{d\sigma}{d\Omega}} = \left| \underbrace{\frac{b}{\sin\theta}}_{\text{kinematics}} \underbrace{\frac{db}{d\theta}}_{\text{Dynamics}} \right|$$

• QM: time-dependent perturbation theory

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H_0 |\psi\rangle \quad |\psi(t)\rangle = \sum_n a_n e^{-iE_n t/\hbar} |n\rangle$$

(unperturbed Hamiltonian) $(H_0 |n\rangle = E_n |n\rangle)$ $\langle a|b\rangle = \int \psi_a^* \psi_b d\vec{x} = \delta_{ab}$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = (H_0 + \varepsilon V) |\psi\rangle$$

$V(\vec{x})$ independent of time but only acting between $t = -T/2$ and $t = +T/2$

$$|\psi(t)\rangle = \sum_n a_n(t) e^{-iE_n t/\hbar} |n\rangle$$

$$i\hbar \sum_n \frac{da_n(t)}{dt} e^{-iE_n t/\hbar} |n\rangle = \varepsilon \sum_n a_n(t) V e^{-iE_n t/\hbar} |n\rangle$$

Multiply by $\langle f|$:

$$i\hbar \frac{da_f(t)}{dt} = \varepsilon \sum_n a_n(t) \underbrace{\langle f|V|n\rangle}_{\int \psi_f^* V(\vec{x}) \psi_n d\vec{x}} e^{+i(E_f - E_n)t/\hbar}$$

Probability $|i\rangle \rightarrow |f\rangle$:

$$P_{i \rightarrow f}(t) = |\langle f|\psi\rangle|^2 = |a_f(t)|^2 \quad (\text{Probability of passing from state } i \text{ to state } f)$$

Write: $a_n(t) = a_n^{(0)}(t) + \varepsilon a_n^{(1)}(t) + \dots$

Interaction during $[-T/2, +T/2]$: $|\psi(t=-T/2)\rangle = |i\rangle$ (initial state)
 $\rightarrow a_n(-T/2) = \delta_{ni}$

• Order 0: $a_f^{(0)}(t) = \text{constant} = \delta_{fi} \Rightarrow \text{probability} = 0$

• Order 1:

$$i\hbar \frac{d}{dt} a_f^{(1)}(t) = \sum_n \langle f|V|n\rangle e^{+i(E_f - E_n)t/\hbar} a_n^{(0)} \quad \leftarrow \text{order 0}$$

$$a_f^{(1)}(t) = \frac{1}{i\hbar} \langle f|V|i\rangle \int_{-T/2}^t dt e^{i(E_f - E_i)t/\hbar}$$

If $T \rightarrow \infty$:

$$A_{fi}^{(1)} = \lim_{T \rightarrow \infty} a_f^{(1)}(T/2) = -2\pi i \langle f|V|i\rangle \delta(E_f - E_i)$$

$$|A_{fi}^{(1)}|^2 = \frac{2\pi}{\hbar} |\langle f|V|i\rangle|^2 \delta(E_f - E_i) \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} e^{i(E_f - E_i)t/\hbar} dt$$

$$= \frac{2\pi}{\hbar} |\langle f|V|i\rangle|^2 \delta(E_f - E_i) \lim_{T \rightarrow \infty} T \quad \text{diverges}$$

$$= \frac{2\pi}{\hbar} |\langle f|V|i \rangle|^2 \delta(E_f - E_i) \lim_{T \rightarrow \infty} T \text{ diverges}$$

Consider instead the transition rate (probability/time)

$$W_{fi} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} |a_f^{(1)}(T)|^2 = \frac{2\pi}{\hbar} |\langle f|V|i \rangle|^2 \delta(E_f - E_i)$$

This is for a single state $|f\rangle$. We want to consider a set of states (with same energy $E_i = E_f$), of which N exist:

$$\rho(E) \equiv \frac{dN}{dE}$$

density of final states

$$W_{fi} \rightarrow \int W_{fi} dN = \int W_{fi} \frac{dN}{dE} dE = W_{fi} \rho(E=E_i)$$

↑
because of $\delta()$

$$W_{fi} = \frac{2\pi}{\hbar} |\langle f|V|i \rangle|^2 \rho(E_i)$$

Fermi Golden rule

(1st order)

dynamics

kinematics

• Order 2:

$$a_f^{(2)}(t) = \frac{1}{i\hbar} \int_{-T/2}^t dt' \sum_n \frac{1}{i\hbar} \left[\langle n|V|i \rangle \int_{-T/2}^{t'} dt'' e^{i(E_n - E_i)t''/\hbar} \right] \langle f|V|n \rangle e^{i(E_f - E_n)t'/\hbar}$$

$\lim_{T \rightarrow \infty}$:

one can obtain a non-divergent expression:

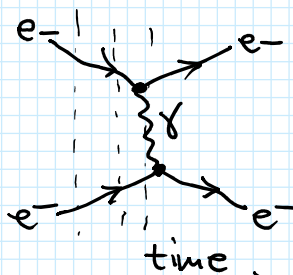
$$\int_{-T/2}^{t'} dt'' e^{i(E_n - E_i)t''/\hbar} \sim \frac{-i\hbar}{E_n - E_i} e^{i(E_n - E_i)t'/\hbar}$$

$$A_{fi}^{(2)} = \frac{2\pi}{i\hbar} \sum_n \langle f|V|n \rangle \frac{1}{E_i - E_n} \langle n|V|i \rangle \delta(E_f - E_i) \quad \left| \begin{array}{l} \text{Two } V \text{ factors and} \\ \text{an Energy factor} \\ 1/(E_i - E_n) \end{array} \right.$$

($E_n \neq E_i$)

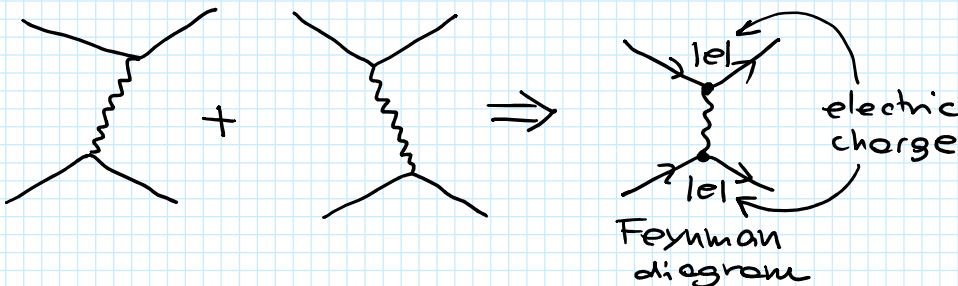
Feynman diagrams
for $\langle f|V|i \rangle$

Example: EM



— e^-
 $\sim \gamma$
 • vertex

Two different time orderings must be added:

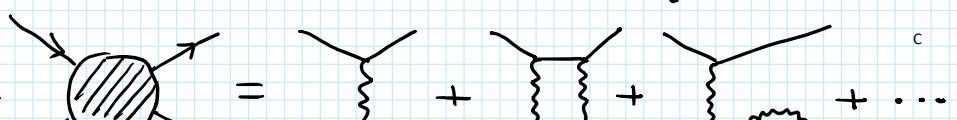


Rule #1: at each vertex

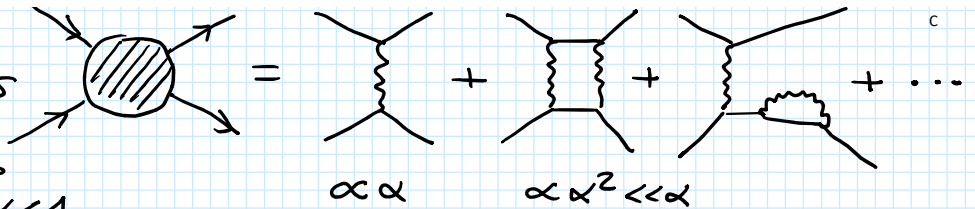
$$\sqrt{\alpha} = \sqrt{\frac{e^2}{4\pi\hbar c}} \approx \sqrt{\frac{1}{137}}$$

(for EM)

• Many diagrams contribute to a single physical process



Contribute to a single physical process



Higher order ones are less important if $\alpha < 1$

- Not all diagrams correspond to physical

Ex. $p_A = p_{A'} + p_\gamma$ $m_A^2 = m_{A'}^2 + 0 + 2p_{A'} \cdot p_\gamma$

In rest system of A' : $p_{A'} = (E_{A'}, \vec{0})$

$2p_{A'} p_\gamma = 2E_{A'} E_\gamma \geq 2m_{A'} E_\gamma \rightarrow E_\gamma = 0$

- In all relativistic theories one has solutions with energies $\pm E$ because $E = \pm \sqrt{p^2 + m^2}$

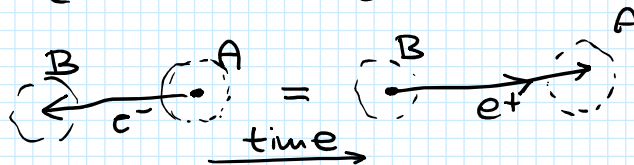
Negative energy particles move backwards in time] interpreted as [positive energy antiparticles which move forward in time]
(Feynman - Stueckelberg)

Mathematically:

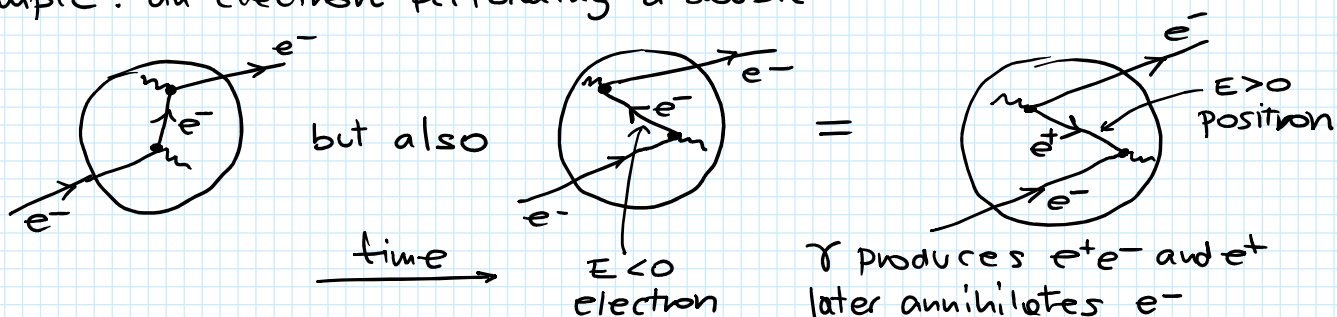
positive energy $e^{-iEt} = e^{-i|E|t}$
negative energy $e^{-i(-|E|)(-t)} = e^{-i|E|t}$

Physically:

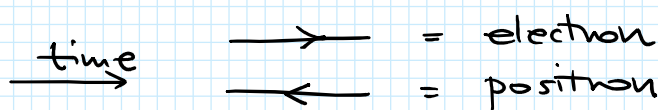
el. charge increases in A and decreases in B



Example: an electron performing a double

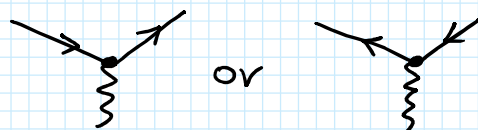


Convention: arrows backward in time are positrons



[Arrows never change directions along a line]

- In electromagnetism the only existing vertex is

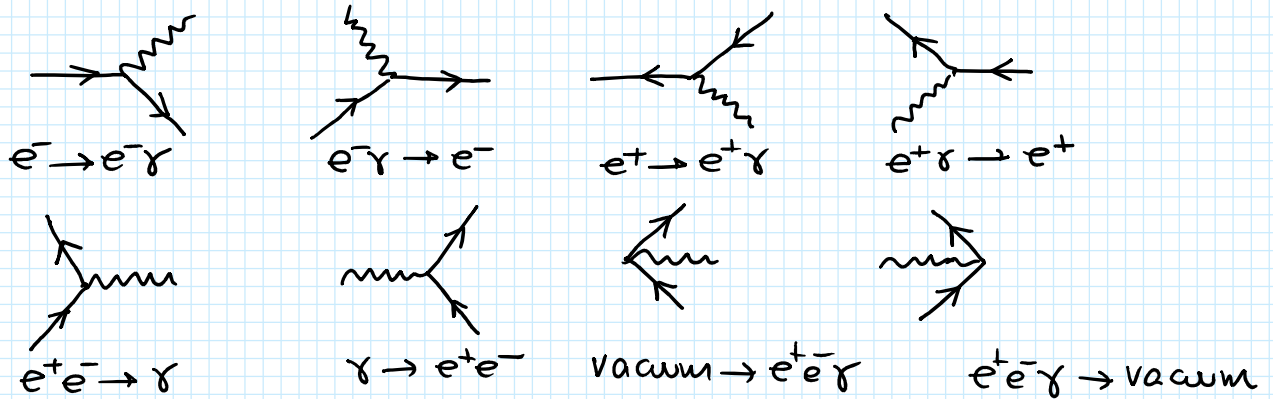


same probability:
 $(-ie)^2 = (ie)^2$
when a second vertex is added

• Crossing symmetry

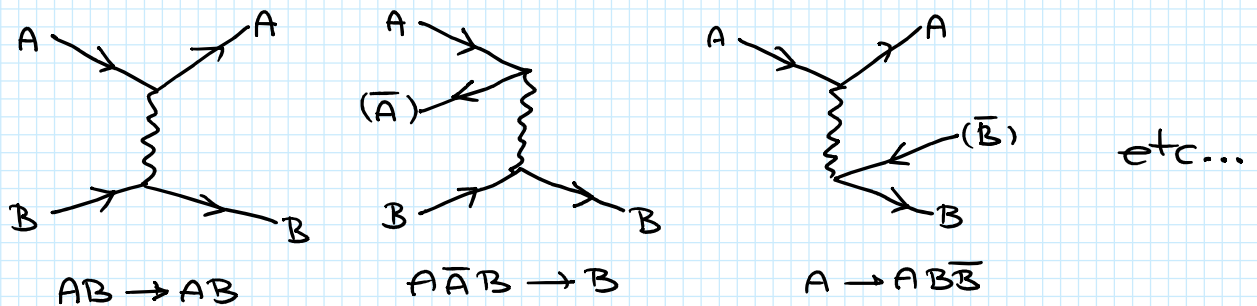
From a diagram, other ones can be obtained by rotation or moving a line from initial to final state or viceversa and simultaneously changing it into its own antiparticle

Ex.



Some of them are kinematically not possible

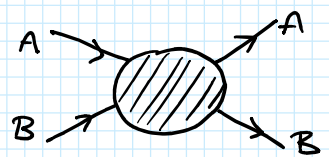
(energy conservation) but otherwise the amplitude is the same



Argument to Justify the Feynman rule for γ in QM

Consider $A(p_A) + B(p_B) \rightarrow A(p'_A) + B(p'_B)$

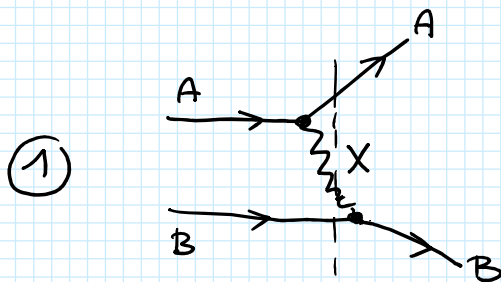
$$p_A + p_B = p'_A + p'_B \text{ of course}$$



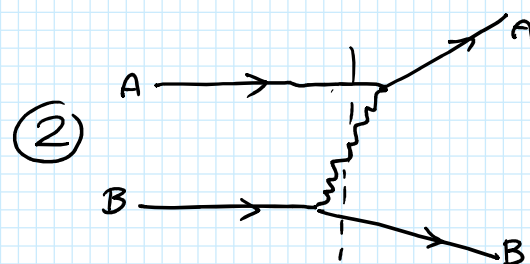
$$E_i = E_A + E_B = E_f = E_{A'} + E_{B'}$$

$$\begin{cases} \Delta \equiv E_A - E_{A'} \\ \vec{q} \equiv \vec{p}_A - \vec{p}_{A'} \end{cases}$$

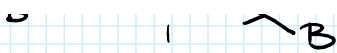
Imagine the lowest-order diagrams with exchange of particle X



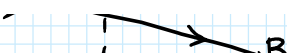
γ was emitted by A



γ was emitted by B

 γ was emitted by A

$$\begin{cases} E_X = E_A - E_A' = \Delta \\ \vec{P}_X = \vec{P}_A - \vec{P}_A' = \vec{q} \end{cases}$$

 γ was emitted by B

$$\begin{cases} E_X = E_B - E_B' = -\Delta \\ \vec{P}_X = \vec{P}_B - \vec{P}_B' = -\vec{q} \end{cases}$$

We can write both as one Feynman diagram but E_X is different in ① and ②; we may imagine energy is not conserved at each vertex: $E_X \neq \Delta, -\Delta$ arbitrary

Energy of intermediate state:

$$\begin{aligned} E_{int} &= E_A' + E_X + E_B \\ &= E_A + E_B - \Delta + E_X \end{aligned}$$

$$\begin{aligned} E_{int} &= E_A + E_X + E_B' \\ &= E_A + E_B + \Delta + E_X \end{aligned}$$

Both possibilities must be added in order 2 perturbation theory (2 vertexes), and each has a factor $\frac{1}{E_i - E_{int}}$ (was " E_n ")

$\langle f|V|i \rangle$ and $\langle u|V|i \rangle$ are equal for ① and ②

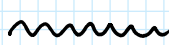
$$(E_i = E_A + E_B)$$

$$\begin{aligned} \left(\frac{1}{E_i - E_{int}} \right)_1 + \left(\frac{1}{E_i - E_{int}} \right)_2 &= \frac{1}{\Delta - E_X} + \frac{1}{-\Delta - E_X} = \frac{2E_X}{\Delta^2 - E_X^2} \\ &= \frac{2E_X}{\Delta^2 - \vec{q}^2 - m_X^2} = \frac{2E_X}{q^2 - m_X^2} \end{aligned}$$

(because $E_X = \vec{P}_X^2 + m_X^2$)

Where four-momentum transfer $q \equiv (\Delta, \vec{q})$

Feynman rule #2

to each  is associated a factor "propagator,"

$$\boxed{\frac{1}{q^2 - m_X^2}}$$

- Note Modern view: both energy and momentum are conserved also at vertexes, but for a virtual particle the mass can be different from m_X i.e. $E_X^2 \neq \vec{P}_X^2 + m_X^2$

$\Rightarrow q$ is the four momentum of X

$\Rightarrow$ Mass of virtual $X = \sqrt{E_X^2 - \vec{P}_X^2}$ arbitrary

• Note For EM where X is a photon ($m_\gamma = 0$)

the propagator is $\frac{1}{q^2}$

the mass of a virtual photon can be anything