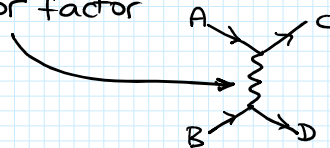


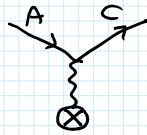
- One more justification for the propagator factor

$$\frac{1}{q^2 - m^2} = \frac{1}{q_0^2 - \vec{q}^2 - m^2}$$

$\hookrightarrow (p_A - p_C)^2$ squared four-momentum transfer



Static potential $V(\vec{r})$
(e.g. very heavy particle
not moving) $E_A = E_C$
($q_0 = 0$)



$$\langle f | V | i \rangle = \int e^{-i\vec{p}_f \cdot \vec{x}} V(\vec{x}) e^{+i\vec{p}_i \cdot \vec{x}} d\vec{x}$$

$$= \int e^{i\vec{q} \cdot \vec{x}} V(\vec{x}) d\vec{x}$$

= Fourier transform of potential ($\vec{q} \equiv \vec{p}_i - \vec{p}_f$)

If $V(\vec{x}) = \frac{1}{|\vec{x}|}$ (Coulomb) $\langle f | V | i \rangle = \frac{4\pi}{q^2}$ ← Recalling that $q^2 = t = -4p^2 \sin^2(\theta^*/2)$ this explains Rutherford $\propto 1/\sin^4(\theta/2)$

If $V(\vec{x}) = \frac{e^{-a|\vec{x}|}}{|\vec{x}|}$ (Yukawa) $\langle f | V | i \rangle = \frac{4\pi}{q^2 + a^2}$ with $a \propto m$

Phase space factor ("kinematic" part of Fermi's formula)

Thomson
ch. 2
Griffiths
ch. 6

- Non relativistic: How many states with $E_f = E_i$?

In QM $\Delta p_x \cdot \Delta x \gtrsim h$ each state with p_x and x within such range is the same state

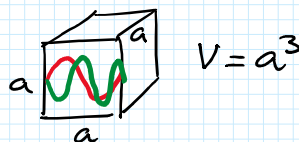
Number of states for a range $\Delta p_x, \Delta x$ $N = \Delta p_x \cdot \Delta x / h$

3 dimensions: $\Delta \vec{p} \cdot \Delta \vec{x} / h^3$

Normalize to 1 particle in some finite volume V (later $V \rightarrow \infty$)

Periodic boundary conditions

$$(p_x, p_y, p_z) = (n_x, n_y, n_z) \frac{2\pi}{a}$$



Integrate over V

$$dN = \frac{d\vec{p} d\vec{x}}{h^3} = \frac{d\vec{p} V}{(2\pi h)^3} \xrightarrow{h=1} \frac{d\vec{p} \cdot V}{(2\pi)^3}$$

phase space element for one particle

Nothing will depend on V (it will cancel) so

Phase space factor for N_f particles ($i=1, 2, \dots, N_f$) in final state:

$$dN = \left(\prod_{i=1}^{N_f-1} \frac{d\vec{p}_i}{(2\pi)^3} \right) = (2\pi)^3 \delta(\vec{P} - \sum_i \vec{p}_i) \prod_{i=1}^{N_f} \frac{d\vec{p}_i}{(2\pi)^3}$$

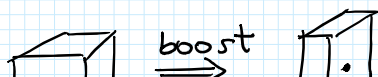
not all $\vec{p}_i$ are independent

$\vec{P}$ = total momentum of state i

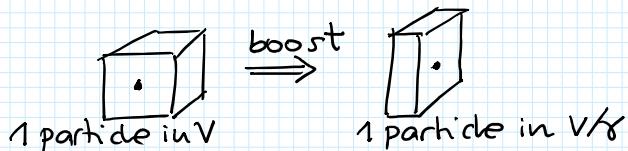
(to be used in formula for $\frac{dN}{dE} = \frac{dN}{dp} \frac{dp}{dE}$)

- Relativistic

Normalization to 1 particle in volume V is not relativistically invariant:



Must normalize to



Must normalize to
 γ particles in volume V
 $\rightarrow E/m$

Conventional choice: $2E$ particles in volume V

New wavefunctions ψ' : $\int_V |\psi'|^2 d\vec{x} = 2E$ $\psi' = \sqrt{2E} \psi$

The matrix element is now

$$\mathcal{M}_{fi} \equiv \langle f' | V | i' \rangle = \sqrt{2E_{i1} 2E_{i2} \dots 2E_{f1} 2E_{f2} \dots} |A_{fi}|$$

Fermi Golden rule becomes:

$$W_{fi} = \frac{2\pi}{(\hbar)} \int |A_{fi}|^2 \delta(E_f - E_i) dN$$

$\xrightarrow{P^0} \sum_i E_i$
 $\xrightarrow{P^0} \sum_i E_i$

$$= \frac{1}{2E_{i1} 2E_{i2} \dots} |M_{fi}|^2 (2\pi)^3 \delta(\vec{P} - \sum_i \vec{p}_i) \frac{d\vec{p}_{f1}}{(2\pi)^3} \frac{d\vec{p}_{f2}}{(2\pi)^3} \dots \frac{1}{2E_{f1} 2E_{f2} \dots}$$

$$W_{fi} = \frac{1}{2E_{i1} 2E_{i2} \dots} |M_{fi}|^2 (2\pi)^4 \delta^4(P - \sum_i p_i) \prod_{i=1}^{N_f} \frac{d\vec{p}_i}{(2\pi)^3 2E_i}$$

$\prod_{i=1}^{N_f} \frac{d\vec{p}_i}{(2\pi)^3 2E_i}$
 dLIPS

For each particle in final state $\frac{d\vec{p}}{(2\pi)^3 2E}$ Lorentz invariant

(Exercise: it can be written as $(2\pi) \delta(p_i^2 - m_i^2) \frac{d^4 p_i}{(2\pi)^4}$)

$N_f = 2$ (two particles in final state)

$$dQ \equiv (2\pi)^4 \delta^4(P - p_1 - p_2) \frac{d\vec{p}_1}{(2\pi)^3 2E_1} \frac{d\vec{p}_2}{(2\pi)^3 2E_2}$$

$$= \frac{1}{(2\pi)^2} \delta(E - E_1 - E_2) \frac{d\vec{p}_1}{2E_1} \frac{1}{2E_2} \quad \text{after integration over } \vec{p}_2$$

Let's evaluate it in CM system: $\begin{cases} |\vec{p}_1^*| = |\vec{p}_2^*| = |\vec{p}_f^*| \\ E_1^* + E_2^* = \sqrt{s} \end{cases}$ (where $E_i^* = p_i^{*2} + m_i^2$)

$$dQ = \frac{1}{(2\pi)^2} \delta(\sqrt{s} - E_1^* - E_2^*) \frac{d\vec{p}_1^*}{2E_1^*} \frac{1}{2E_2^*}$$

$$d\vec{p}_1^* = p_f^{*2} d|\vec{p}_f^*| d\Omega^* = p_f^{*2} \frac{d|\vec{p}_f^*|}{d\sqrt{s}} d\sqrt{s} d\Omega^* = |\vec{p}_f^*|^2 \frac{E_1^* E_2^*}{|\vec{p}_f^*| \sqrt{s}} d\sqrt{s} d\Omega^*$$

$$dQ = \frac{1}{(2\pi)^2} \frac{1}{4E_1^* E_2^*} |\vec{p}_f^*| \frac{E_1^* E_2^*}{\sqrt{s}} d\Omega^* = \frac{1}{(4\pi)^2} \frac{|\vec{p}_f^*|}{\sqrt{s}} d\Omega^*$$

after integration over $\sqrt{s}$ (constant)

Two-body decay $A \rightarrow BC$ ($\sqrt{s} = m_A$)

$$d\Gamma = (\hbar) W_{fi} = \frac{|M|^2}{2m_A} dQ = \frac{|\vec{p}_f^*|}{32\pi^2 m_A^2} |M|^2 d\Omega^*$$

decay width decay rate

Two examples:

$\phi \rightarrow e^+ e^-$ $BR_\phi = 2.97 \cdot 10^{-4}$ Γ_μ BR_μ

Two examples:

$$\textcircled{1} \quad \begin{aligned} \phi &\rightarrow e^+e^- & BR_e &= 2.97 \cdot 10^{-4} \\ \phi &\rightarrow \mu^+\mu^- & BR_\mu &= 2.86 \cdot 10^{-4} \end{aligned} \quad \frac{\Gamma_\mu}{\Gamma_e} = \frac{BR_\mu}{BR_e} = 0.961$$

$$(m_\phi = 1020 \text{ MeV})$$

We have $\frac{|\vec{p}_\mu^*|}{|\vec{p}_e^*|} = \frac{\sqrt{(m_\phi/2)^2 - m_\mu^2}}{\sqrt{(m_\phi/2)^2 - m_e^2}} \approx \sqrt{1 - \frac{4m_\mu^2}{m_\phi^2}} = 0.978$

$$\left(\begin{array}{l} m_e \approx 0.511 \text{ MeV} \\ m_\mu \approx 106 \text{ MeV} \end{array} \right) \quad \begin{array}{l} \Gamma_\mu/\Gamma_e \text{ is practically described by the ratio} \\ \text{of phase space factors} \Rightarrow |\mathcal{M}|_\mu^2 \approx |\mathcal{M}|_e^2 \end{array} \quad (m_e \ll m_\mu, m_\phi)$$

$$\textcircled{2} \quad \begin{aligned} \pi^- &\rightarrow e^- \bar{\nu}_e & BR_e &\approx 1.2 \cdot 10^{-4} \\ \pi^- &\rightarrow \mu^- \bar{\nu}_\mu & BR_\mu &\approx 99.988 \cdot 10^{-2} \end{aligned}$$

$$(m_\pi = 139 \text{ MeV})$$

$$\frac{|\vec{p}_\mu^*|}{|\vec{p}_e^*|} \approx \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 - m_e^2} \approx 1 - \frac{m_\mu^2}{m_\pi^2} = 0.43$$

(m_\nu \approx 0) (m_e \ll m_\mu, m_\pi)

Very different from $\Gamma_\mu/\Gamma_e \approx 8300$!

$\Rightarrow$ In this case $|\mathcal{M}|_\mu^2 \gg |\mathcal{M}|_e^2$

Two-body cross section $AB \rightarrow CD$

Here we want to compute $\sigma = W_{fi} / \underbrace{\Phi_A \cdot N_B}_{\text{Luminosity}} \leftarrow \text{beam flux}$

$\leftarrow \text{number of scattering centres}$

$\underbrace{\vec{A} \rightarrow \vec{B}}_{\text{(beam)}} \quad \underbrace{\cdots B}_{\text{(target)}}$

$$\Phi_A = \frac{\text{number of A}}{\text{area time}} = \frac{\text{density of A} \cdot \text{volume}}{\text{area time}} = \frac{n_A \cdot S \cdot (v_A + v_B) \Delta t}{S \Delta t}$$

$\underbrace{(\underbrace{S}_{\text{area}} \cdot \underbrace{(v_A + v_B) \Delta t}_{\text{length}}))}_{\text{volume}}$

$$n_A = \frac{1}{V} \text{ non relativistically}$$

$$\Phi_A = (v_A + v_B)/V$$

(V will cancel out)

Consider the case of a single target particle $N_B = 1$

But relativistically we have also $\frac{1}{2E_A} \frac{1}{2E_B}$ so:

Relativistically invariant flux factor

$$\Phi_A N_B \rightarrow \boxed{F = 4E_A E_B (v_A + v_B)}$$

(Exercise: it can be written as $F = \sqrt{(p_A p_B)^2 - m_A^2 m_B^2}$)

In CM system:

$$F = 4E_A^* E_B^* \left(\frac{|\vec{p}_A^*|}{E_A^*} + \frac{|\vec{p}_B^*|}{E_B^*} \right) = 4|\vec{p}_i^*| (E_A^* + E_B^*) = 4|\vec{p}_i^*| \sqrt{s}$$

$$(|\vec{p}_A^*| = |\vec{p}_B^*| = |\vec{p}_i^*|)$$

$$d\sigma = \frac{1}{F} |\mathcal{M}|^2 d\Omega = \frac{1}{4|\vec{p}_i^*| \sqrt{s}} |\mathcal{M}|^2 \frac{1}{(4\pi)^2} \frac{|\vec{p}_f^*|}{\sqrt{s}} d\Omega^*$$

$$d\sigma = \frac{1}{G4\pi^2 s} \frac{|\vec{P}_f|}{|\vec{P}_i|} |\mathcal{M}|^2 d\Omega^*$$

DIRAC EQUATION

Thomson ch. 4
Griffiths ch. 7

Relativistically $E^2 = \vec{p}^2 + m^2$ gives in quantum mechanics an equation such as Klein-Gordon

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0$$

From $\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} = \frac{\partial}{\partial t} \left(\underbrace{\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}}_{-i\rho} \right) = \nabla \cdot \underbrace{(\psi^* \nabla \psi - \psi \nabla \psi^*)}_{i\vec{j}}$

one identifies:

$$\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \right)$$

and for a plane wave $\psi = N e^{i(\vec{p} \cdot \vec{x} - Et)}$

one obtains $\rho = 2/|N|^2 E \geq 0$

Since the problem is due to the second time derivative Dirac looked for an equation of first order:

$$E\psi = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)\psi \quad i\hbar \frac{\partial \psi}{\partial t} = H\psi = E\psi$$

or $i\hbar \frac{\partial \psi}{\partial t} = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi$ where $\vec{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$

This must still be consistent with $E^2 = \vec{p}^2 + m^2$ so squaring

$$-\frac{\partial^2 \psi}{\partial t^2} = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi \quad \text{must be} \quad = -\nabla^2 \psi + m^2 \psi$$

This requires

$$\begin{cases} \alpha_i^2 = \beta^2 = 1 \\ \alpha_i \beta + \beta \alpha_i = 0 \\ \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad (i \neq j) \end{cases}$$

α_i, β cannot be numbers but can be matrices

$\Rightarrow \psi$ is a column vector, more than one wavefunction

Properties of these matrices:

1) $\text{Tr}(\alpha_i) = \text{Tr}(\alpha_i \beta \beta) = \text{Tr}(\beta \alpha_i \beta) = -\text{Tr}(\beta \beta \alpha_i) = -\text{Tr}(\alpha_i) = 0$

2) $\alpha_i X = \lambda X \quad X = \lambda \alpha_i X = \lambda^2 X \quad \lambda = \pm 1 \quad \text{eigenvalues}$

Since Trace = $\sum$ eigenvalues $\rightarrow$ matrices of even order

3) Hermitian (so that H is Hermitian)

2x2 matrices? There are only 3 non-commuting ones
(Pauli matrices) we need 4 ($\alpha_1, \alpha_2, \alpha_3, \beta$)

$\Rightarrow$ 4x4 matrices

Dirac matrices

$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ Pauli

$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ 2x2 unity

e.g. $\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ etc.

$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Other choices are possible (change of basis)