

$$\psi^\dagger = \psi^*{}^T$$

$$= i 2 \psi^\dagger \cdot \alpha_x + \dots$$

$$\frac{\partial p}{\partial t} + \nabla \cdot \vec{J} = 0 \quad \text{with}$$

$$\rightarrow |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0$$

$$H = \vec{\alpha} \cdot \vec{p} + \beta m$$

Spin

$$[H, \vec{\sigma}] = +i \vec{\alpha} \times \vec{p}$$

$$\vec{J} = \vec{S} + \vec{L} \quad \text{total angular momentum conserved}$$

$$S^2 | \psi \rangle = S(S+1) | \psi \rangle$$

→ $s = \frac{1}{2}$
Spin $\frac{1}{2}$

Magnetic moment charge
 $\vec{\mu} = \frac{q}{m} \vec{S}$
 mass

In general $\vec{\mu} = g \frac{q}{2m} \vec{S}$

and for Dirac $g=2$

as experimentally measured

$$\left. \begin{aligned} \gamma^0 &\equiv \beta \\ \gamma^i &\equiv \beta \alpha_i \end{aligned} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (i=1,2,3)$$

$$\gamma^\mu \equiv (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

$$(i\gamma^0 \partial_0 + i\vec{\gamma} \cdot \vec{\nabla} - m)\psi = 0 \Rightarrow (i\gamma^\mu \partial_\mu - m)\psi$$

$$\begin{cases} \gamma_0^2 = 1 & \gamma_0^\dagger = \gamma_0 \\ \gamma_i^2 = -1 & \gamma_i^\dagger = -\gamma_i \end{cases} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad [\{a, b\} \equiv ab + ba]$$

$$= \text{dist} = d^{\top} x_{\text{max}} d$$

$$|0\rangle = 1$$

$$\rho = \psi^\dagger \psi = \psi^\dagger \gamma_0 \psi$$

$$\vec{j} = \psi^\dagger \vec{\alpha} \psi = \psi^\dagger \gamma_0 \vec{\gamma} \psi$$

$$J^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma_0 \gamma^\mu \psi = \bar{\psi} \gamma^\mu \psi$$

$$\text{where } \bar{\psi} \equiv \psi^\dagger \gamma_0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

Solutions

$$\psi(\vec{x}, t) = u(E, \vec{p}) e^{i(\vec{p} \cdot \vec{x} - Et)}$$

plane wave

$$(\gamma^\mu p_\mu - m) u(E, \vec{p}) = 0$$

column four-vector
(Dirac spinor)

$$u = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{E=+m} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{E=-m} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{E=+m} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{E=-m}$$

• At rest $\vec{p}=0$: $E \gamma^0 u = m u$

• Arbitrary $\vec{p}$: $(E \gamma^0 - \vec{p} \cdot \vec{\gamma} - m) u = 0$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] u = 0$$

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}$$

$$\begin{pmatrix} (E-m) \mathbb{1} & -\vec{\sigma} \cdot \vec{p} \\ +\vec{\sigma} \cdot \vec{p} & -(E+m) \mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

two-component spinors

$$u_A = \frac{\vec{\sigma} \cdot \vec{p}}{E-m} u_B$$

$$u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u_A$$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}$$

$E > 0$

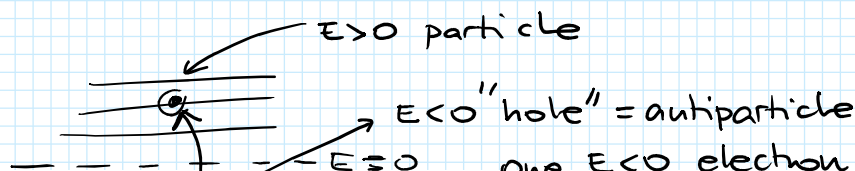
$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + i p_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - i p_y}{E-m} \\ -\frac{p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$E < 0$
 $E = -\sqrt{\vec{p}^2 + m^2}$

$$u_i^\dagger u_j = 0 \text{ if } i \neq j$$

Interpretation of $E < 0$ solutions

"Dirac sea" of
 $E < 0$ electrons is



A diagram showing horizontal energy levels. A dashed line represents $E = 0$. Below this line, there are several levels with dots representing electrons. One level below $E = 0$ has a dot that is circled. An arrow points from this circled dot to the text on the right. The text explains that this represents an $E < 0$ "hole" or antiparticle, which is equivalent to one $E < 0$ electron missing from the sea, or $\leftrightarrow$ 1 antiparticle ($E > 0$).

$E < 0$ "hole" = antiparticle
 one $E < 0$ electron missing from sea
 $\leftrightarrow$ 1 antiparticle ($E > 0$)

$$\begin{cases} v_1(E, \vec{p}) e^{-i(\vec{p} \cdot \vec{x} - Et)} = u_4(-E, -\vec{p}) e^{+i(-\vec{p} \cdot \vec{x} - (-E)t)} \\ v_2(E, \vec{p}) \quad " \quad " = u_3(-E, -\vec{p}) \quad " \quad " \end{cases}$$

$$\underline{(\gamma^\mu p_\mu + m) V = 0} \quad \text{f.} \quad V = \begin{pmatrix} V_A \\ V_B \end{pmatrix}$$

$$V_A = \frac{\overline{\sigma \cdot P}}{E + m} V_B$$

$$V_B = \frac{\overline{F \cdot P}}{F - m} V_A$$

$$v_1 = \overline{N_1} \begin{pmatrix} \frac{p_x - i p_y}{E + m} \\ -\frac{p_z}{E + m} \\ 0 \\ 1 \end{pmatrix}$$

$$V_2 = \overline{N}_2 \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

with

$$E = +\sqrt{p^2 + m^2}$$

(also r_3, r_4 with $E < 0$)

$$\int_V \psi^\dagger \psi \, dx = 2E \quad \text{in relativity}$$

$$u_1^\dagger u_1 = |N_1|^2 \frac{2E}{E+m} \rightarrow N_1 = \sqrt{E+m}$$

for all u_1, u_2, v_1, v_2

$$S_Z = \frac{1}{2} \Sigma_Z = \frac{1}{2} \begin{pmatrix} \sigma_Z & 0 \\ 0 & \sigma_Z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$u_1(m, \vec{0}) = N_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} : S_z u_1(m, \vec{0}) = \frac{1}{2} u_1(m, \vec{0}) \quad \text{eigenstate of } S_z$$

but for $\vec{p} \neq 0$ u_1, u_2, v_1, v_2 are not eigenstates of S_z

However if $\vec{p} = \pm p \hat{z}$: $u_1 \propto \begin{pmatrix} 1 \\ 0 \\ \pm p \\ E+m \\ 0 \end{pmatrix}$ $u_2 \propto \begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp p \\ E+m \end{pmatrix}$

$$\begin{cases} S_z u_1(E, \pm p \hat{z}) = +\frac{1}{2} u_1(\dots) \\ S_z u_2(\dots) = -\frac{1}{2} u_2(\dots) \end{cases} \quad \text{eigenstates of } S_z$$

$$S_z u_2(\dots) = -\frac{1}{2} u_2(\dots)$$

Antiparticles: $\text{spin}(\text{observed}) = -\frac{1}{2} \vec{\Sigma}$ (Just as for $E, \vec{P}$)

$$S_z(\text{observed}) v_1(E, \pm p \hat{z}) = +\frac{1}{2} v_1(\dots)$$

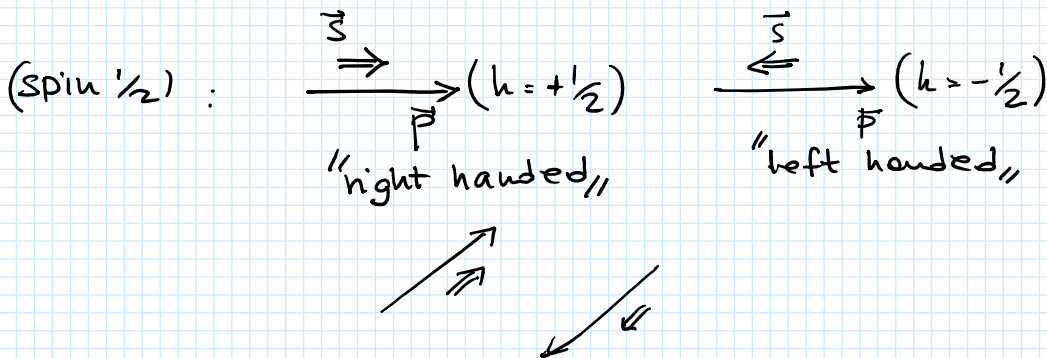
$$S_z \dots v_2(\dots) = -\frac{1}{2} v_2(\dots)$$

$$h = \frac{\vec{S} \cdot \vec{P}}{P}$$

helicity

$$h = \frac{\vec{\Sigma} \cdot \vec{P}}{2P} = \frac{1}{2P} \begin{pmatrix} \vec{\sigma} \cdot \vec{P} & 0 \\ 0 & \vec{\sigma} \cdot \vec{P} \end{pmatrix}$$

$[H, h] = 0$ conserved



Helicity not relativistically invariant
(except if $m=0$)

$$\bar{\psi} = \psi^\dagger \gamma_0$$

$$\bar{u} = u^\dagger \gamma_0$$

Feynman rules for QED

initial particle $\longrightarrow \bullet$ $u(p)$

final particle $\bullet \longrightarrow$ $\bar{u}(p)$

initial antiparticle $\longleftarrow \bullet$ $\bar{v}(p)$

final antiparticle $\bullet \longleftarrow$ $v(p)$

[Arrows never change direction]

Vertex

$$-i \Phi |e| \gamma^\mu$$

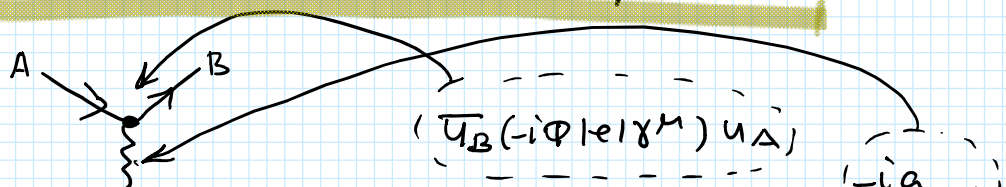
($\Phi = -1$ for electron)

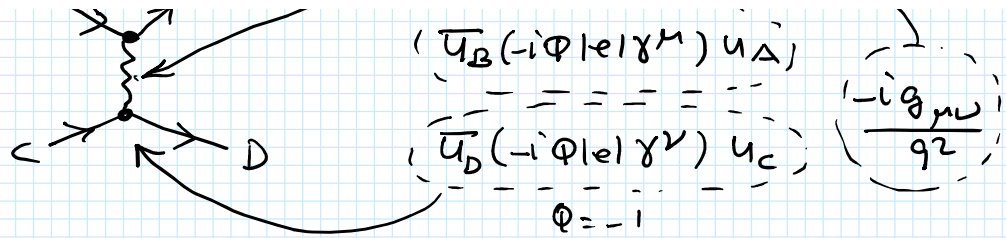
($\Phi = +1$)

(virtual)

Photon propagator

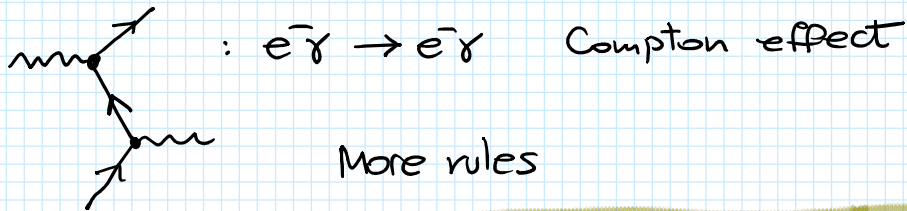
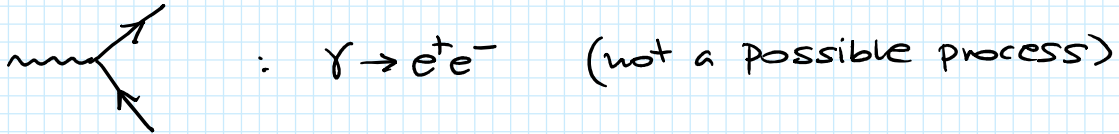
$$\frac{1}{q^2} (-i g_{\mu\nu})$$



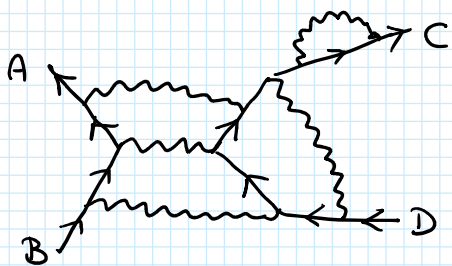


$$-i\mathcal{M} = \bar{u}_B(i|e|\gamma^\mu) u_A \frac{(-i)}{q^2} \bar{u}_D(+i|e|\gamma_\mu) u_C$$

$$|\mathcal{M}|^2 \propto \frac{e^4}{q^4} \rightarrow \text{Rutherford}$$



initial photon		$\epsilon^\mu(p)$	photon polarization vector (like A^μ)
final photon		$\epsilon^{*\mu}(p)$	
virtual electron		$-i \frac{\gamma^\mu q_\mu + m}{q^2 - m_e^2}$	(virtual) electron propagator



All internal momenta which are not fixed by p_A, p_B, p_C, p_D are integrated over