

Helicity spinors

$$[H, h] = 0$$

Recall: $h = \frac{\vec{S} \cdot \vec{p}}{p} = \frac{1}{2p} \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$

$$hu = \lambda u \quad u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p}) u_A &= 2p\lambda u_A \\ (\vec{\sigma} \cdot \vec{p}) u_B &= 2p\lambda u_B \end{aligned} \xrightarrow{(\vec{\sigma} \cdot \vec{p})^2 = p^2} p^2 u_A = 2p\lambda \vec{\sigma} \cdot \vec{p} u_A = 4p^2 \lambda^2 u_A$$

(check as exercise) $\Rightarrow \lambda = \pm 1/2$

Recall $(\vec{\sigma} \cdot \vec{p}) u_A = (E+m) u_B \Rightarrow u_B = 2\lambda \frac{p}{E+m} u_A$

For motion in a generic direction

$$\vec{p} = (p \sin\theta \cos\phi, p \sin\theta \sin\phi, p \cos\theta)$$

$$\frac{1}{2p} \vec{\sigma} \cdot \vec{p} = \frac{1}{2} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

writing $u_A = \begin{pmatrix} a \\ b \end{pmatrix} \quad \frac{b}{a} = \frac{2\lambda - \cos\theta}{\sin\theta} e^{i\phi}$

For $\lambda = +1/2$: $\frac{b}{a} = \frac{1 - \cos\theta}{\sin\theta} e^{i\phi} = \frac{2 \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} e^{i\phi} = \tan(\theta/2) e^{i\phi}$

$$u_+ = u(\lambda = +1/2) = N \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \\ \frac{p}{E+m} \cos(\theta/2) \\ \frac{p}{E+m} \sin(\theta/2) e^{i\phi} \end{pmatrix}$$

$\sqrt{E+m}$

$\vec{S} \Rightarrow \vec{p}$

$u_- = u(\lambda = -1/2) = \dots$ (exercise)

For antiparticles ($\bar{v}$ spinors) the spin is opposite to the one given by $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, so

$$\frac{\vec{\Sigma} \cdot \vec{p}}{2p} v(\lambda = +1/2) = -\frac{1}{2} v(\lambda = +1/2)$$

$$v_+ = v(\lambda = +1/2) = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} \sin(\theta/2) \\ -\frac{p}{E+m} \cos(\theta/2) e^{i\phi} \\ -\sin(\theta/2) \\ \cos(\theta/2) e^{i\phi} \end{pmatrix} \quad v_- = v(\lambda = -1/2) = \dots$$

(exercise)

- In the ultra-relativistic limit $E \gg m$

$$u(\lambda = +1/2) \longrightarrow \sqrt{E} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \\ \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix} \quad \text{etc.}$$

Chirality

Eigenvalue of $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$$\begin{cases} \gamma_5^2 = \mathbb{1} \\ \gamma_5^\dagger = \gamma_5 \\ \gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5 \end{cases}$$

$$\begin{cases} \gamma_5 u_R = +u_R & \text{("right-handed")} \\ \gamma_5 u_L = -u_L & \text{("left-handed")} \end{cases}$$

$$\begin{cases} \gamma_5 v_R = +v_R \\ \gamma_5 v_L = -v_L \end{cases}$$

[Note: different convention in Thomson]

Projectors $\begin{cases} P_R = \frac{1}{2}(1 + \gamma_5) \\ P_L = \frac{1}{2}(1 - \gamma_5) \end{cases} \quad P_R + P_L = 1$

$$u = u_R + u_L$$

$$\begin{aligned} P_R u &= u_R \\ P_L u &= u_L \end{aligned}$$

$$\begin{aligned} P_R P_R &= P_R \\ P_L P_L &= P_L \end{aligned}$$

$$P_R P_L = P_L P_R = 0$$

- $[H, \gamma_5] \neq 0$: chirality is not a constant of motion

$$H = \vec{\alpha} \cdot \vec{p} + \beta m = \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m$$

$\uparrow$ commutes with γ_5 $\nwarrow$ does not commute

$$\text{so if } m=0 \quad [H, \gamma_5] = 0$$

chirality is Lorentz-invariant

- $u_L = \frac{1}{2}(1 - \gamma_5) u$

$$\bar{u}_L \equiv u_L^\dagger \gamma^0 = u^\dagger \frac{1}{2}(1 - \gamma_5^\dagger) \gamma^0 = u^\dagger \frac{1}{2}(1 - \gamma_5) \gamma^0 = u^\dagger \gamma^0 \frac{1}{2}(1 + \gamma_5) = \bar{u} \frac{1}{2}(1 + \gamma_5)$$

$$\bar{u}_L u_L = \frac{1}{4} \bar{u} (1 + \gamma_5) (1 - \gamma_5) u = 0 \quad \bar{u}_R u_R = 0$$

Relation to helicity

In the limit $E \gg m$ (neglect m)

$$\left[\begin{array}{l} u(\lambda=+1/2) \rightarrow u_R \\ u(\lambda=-1/2) \rightarrow u_L \\ v(\lambda=+1/2) \rightarrow v_L \\ v(\lambda=-1/2) \rightarrow v_R \end{array} \right] \text{--- (opposite convention in Thomson)}$$

	helicity	chirality
const. of motion	✓	✗ (✓ if $m=0$)
Lorentz-invariant	✗ (✓ if $m=0$)	✓

• The EM interaction is $J^\mu A_\mu = \bar{\psi} \gamma^\mu \psi A_\mu$
 contains $\bar{u}_L + \bar{u}_R$ ← contains $u_L + u_R$

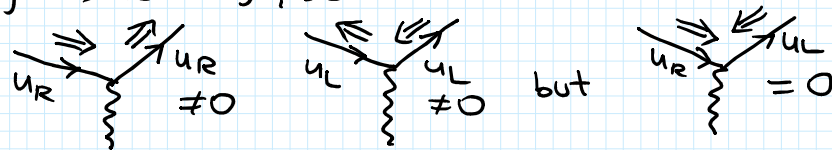
$$\bar{u}_L \gamma^\mu u_R = \frac{1}{4} \bar{u} (1 + \gamma_5) \gamma^\mu (1 + \gamma_5) u = \frac{1}{4} \bar{u} (1 + \gamma_5) (1 - \gamma_5) \gamma^\mu u = 0$$

$$\bar{u}_R \gamma^\mu u_L = 0$$

only $\bar{u}_L \gamma^\mu u_L$ and $\bar{u}_R \gamma^\mu u_R$ are $\neq 0$

If $E \gg m$ chirality $\rightarrow$ helicity, so this indicates that

at high energies



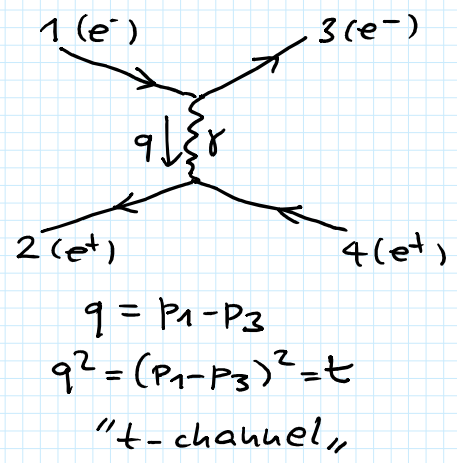
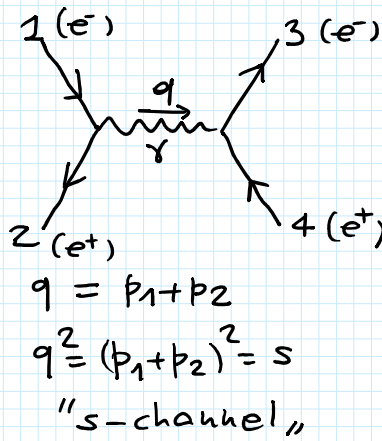
helicity is conserved at each EM interaction vertex

Some QED processes

• $e^+ e^- \rightarrow e^+ e^-$

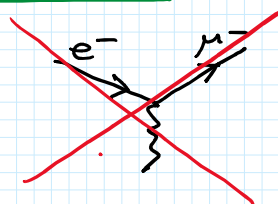
Two possible diagrams

$e^+ e^- \rightarrow e^+ e^-$



• $e^+ e^- \rightarrow \mu^+ \mu^-$

Simpler, only s-channel

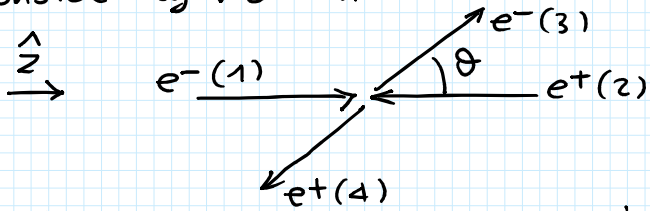


The EM vertex has the same fermion on both lines
 it does not transform one particle in another

Exercise [i.e. Thomson section 6.2]

Compute $|\mathcal{M}|^2$ for the process $e^+ e^- \rightarrow \mu^+ \mu^-$ (for $E \gg m$)
 considering the different helicity amplitudes (in the center of mass)

Compute $|M|$ for the process $e^- e^+ \rightarrow \mu^- \mu^+$ (for $e^- \rightarrow \mu^-$), considering the different helicity amplitudes (in the center of mass)



Each fermion can have two helicities $\Rightarrow$ $\begin{matrix} u_+ & u_- \\ v_+ & v_- \end{matrix}$ $\begin{matrix} \Leftarrow \\ \Leftarrow \end{matrix}$ $\begin{matrix} (= u_R, u_L) \\ (= v_L, v_R) \end{matrix}$

4 possible initial states
4 possible final states

$\Rightarrow$ 16 possible amplitudes (distinguishable)

$++ \rightarrow ++$ $++ \rightarrow +- \dots +- \rightarrow ++$ $+- \rightarrow +- \dots$

Initial state not polarized = same number of R and L

$$\langle |M|^2 \rangle = \frac{1}{4} [|M|_{++ \rightarrow ++}^2 + |M|_{++ \rightarrow +-}^2 + \dots]$$

4 possible initial states

Write $u_{+,-}(1)$ ($\theta=0, \varphi=0$) $v_{+,-}(2)$ ($\theta=\pi, \varphi=0$)
 $u_{+,-}(3)$ (any θ, φ) $v_{+,-}(4)$ (any θ, φ)

$$\text{write } M_{++ \rightarrow ++} = \bar{u}_+(3) \gamma^\mu v_+(4) \frac{e^2}{q^2} \bar{v}_+(2) \gamma_\mu u_+(1) \quad \text{etc.}$$

take $|M|^2$

sum

Only $M_{+- \rightarrow +-}$, $M_{+- \rightarrow -+}$, $M_{-+ \rightarrow -+}$, $M_{-+ \rightarrow +-}$ are $\neq 0$

$$\langle |M|^2 \rangle = e^4 (1 + \cos^2 \theta)$$

Use formula for cross-section into 2 particles

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)$$

ISOSPIN

- $m_p \approx 938 \text{ MeV} \approx m_n \approx 939 \text{ MeV}$
- Strong interaction similar for pp , pn , nn

Idea that p and n are two states of a nucleon N

and strong interactions consider the two states as equal

$$N = (p, n) \quad p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{"isotopic spin or isospin"}$$

Analogous to spin $\frac{1}{2}$

$$\vec{I} = (I_1, I_2, I_3) \quad I = \frac{1}{2} \quad I_3 = \pm \frac{1}{2}$$

$$+ - (+1, +2, +3)$$

$$|I, I_3\rangle = \begin{cases} |1/2, 1/2\rangle = |p\rangle \\ |1/2, -1/2\rangle = |n\rangle \end{cases}$$

Rotations in isospin space: $U = e^{i\vec{\alpha} \cdot \vec{I}}$ (2x2 matrix) $\vec{I}$ Hermitian
8 real parameters

$$U^\dagger U = \mathbb{1} \quad (4 \text{ conditions})$$

→ 4 free parameters

- $\det(U^\dagger U) = \det(\mathbb{1}) = +1$

$$= \det(U^\dagger) \det(U) = \det(U^*) \det(U) = \det(U)^* \det(U) = |\det(U)|^2$$

but for no rotation ($\vec{\alpha} = 0$) $U = \mathbb{1}$ with $\det(U) = +1$

so $\det(U) = +1$

- One of the four parameters is in $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$ which only changes the phase

The other 3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\text{If } U = e^A \quad \det(U) = e^{\text{tr}(A)} \quad \text{so } \text{tr}(A) = 0 \right]$$

$$\boxed{\vec{I} = \frac{1}{2} \vec{\sigma}} \quad \text{so that } I_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\sigma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \sigma_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \sigma_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \dots$$

$$\boxed{I_\pm \equiv I_1 \pm iI_2} = \frac{1}{2} \sigma_1 \pm i\sigma_2 =$$

$$I_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad I_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \dots$$

$$\boxed{[I_i, I_j] = i\epsilon_{ijk} I_k}$$

i.e. $[I_1, I_2] = iI_3$ etc.

$$I^2 = I_1^2 + I_2^2 + I_3^2 \quad \text{is Hermitian and } [I^2, I_i] = 0$$

$$I^2 = I_- I_+ + I_3 + I_3^2 \quad I^2 |I, +I\rangle = (0 + I + I^2) |I, +I\rangle = I(I+1) |I, +I\rangle$$

Since $[I^2, I_-] = 0$ all other $|I, I_3\rangle$ have same I

$$\begin{cases} I^2 |I, I_3\rangle = I(I+1) |I, I_3\rangle \\ I_3 |I, I_3\rangle = I_3 |I, I_3\rangle \quad (-I \leq I_3 \leq I) \end{cases}$$

$$\begin{cases} I_3 |I, I_3\rangle = I_3 |I, I_3\rangle & (-I \leq I_3 \leq I) \\ I_{\pm} |I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)} |I, I_3 \pm 1\rangle \end{cases}$$

For $I = 1/2$: $I_+ |n\rangle = |p\rangle$ $I_+ |p\rangle = 0$
 $I_- |p\rangle = |n\rangle$ $I_- |n\rangle = 0$

• Composition of isospin

$$\begin{aligned} I^{(1)}, I^{(2)} &\rightarrow I, I_3 \\ \begin{cases} |I^{(1)} - I^{(2)}| \leq I \leq I^{(1)} + I^{(2)} \\ I_3 = I_3^{(1)} + I_3^{(2)} \end{cases} \end{aligned}$$

$$|pp\rangle = |1/2, 1/2\rangle |1/2, 1/2\rangle = |1, +1\rangle$$

$$\begin{aligned} I_- |pp\rangle &= |1/2, -1/2\rangle |1/2, +1/2\rangle + |1/2, 1/2\rangle |1/2, -1/2\rangle = |n\rangle |p\rangle + |p\rangle |n\rangle \\ &= \sqrt{1(2) - 1(0)} |1, 0\rangle = \sqrt{2} |1, 0\rangle \end{aligned}$$

$$\text{so } |1, 0\rangle = \frac{1}{\sqrt{2}} (|p\rangle |n\rangle + |n\rangle |p\rangle)$$

$$I_- |1, 0\rangle = |1, -1\rangle = |n\rangle |n\rangle$$

$$\text{And there is also } |0, 0\rangle = \frac{1}{\sqrt{2}} (|p\rangle |n\rangle - |n\rangle |p\rangle)$$

Notation $2 \otimes 2 = 3 \oplus 1$

2 states ($I = 1/2$) $\nearrow$ $\nwarrow$ 3 states ($I = 1$) $\nwarrow$ 1 state ($I = 0$)

$$\text{Similarly } 3 \otimes 2 = 4 \oplus 2$$

$$\text{isospin } 1 \otimes \text{isospin } 1/2 \rightarrow \text{isospin } 3/2 + \text{isospin } 1/2$$

• Isospin symmetry

if only strong interactions
 if $m_p = m_n$

gives

(a) relations between $\mathcal{H}$ for states with "same particles" in different isospin $\swarrow$ (strong)

(b) classification of hadrons $\rightarrow$ quark model