FUNDAM. INT.

Wednesday, October 28, 2026 10:44 AM

We built baryons from quarks (qqq)

To build mesons ($q\bar{q}$) we must know how $\bar{q}$ transform

$q = \begin{pmatrix} u \\ d \end{pmatrix}^{+1/2}_{-1/2}$ transform as $\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix}$ $\quad q' = Uq$

$\quad\quad\quad

u } isospin
d }
s ← strangeness

Group theory

- closure
- associativity
- ∃ identity
- ∃ inverse

We are interested in groups of transfomations

- **Abelian** = all transfomations commute
- Similarity transfomation $g \to S g S^{-1} = g'$ (group elements)
  is such that
  if $g|a\rangle = |b\rangle$
  one has $g'|a'\rangle = |b'\rangle$ by choosing $\begin{cases} |a'\rangle \equiv S|a\rangle \\ |b'\rangle \equiv S|b\rangle \end{cases}$
- Subgroup ($F \subset G$)
- Direct product $F \otimes G$
- Continuous groups = **Lie groups**
  have parameters (real) α_a $(a = 1, 2, \dots, N)$ → dimension of the group
  α_a can take $\begin{cases} \text{infinite set of values} \\ \text{finite set of values} \quad \text{(compact group)} \end{cases}$
- **Simple** group = does not contain invariant subgroups
  (i.e. subgroups whose elements only transform among themselves)

simple and compact Lie groups : 5 classes : $SU(n)$, $SO(n)$ + others

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- **Unitary group** $U(n)$ complex matrices $n \times n$ such that $U^+ = U^{-1}$
  $2n^2$ real parameters $- n^2$ conditions $= n^2$ (dimension)
  $U(1) = e^{i\phi}$ (complex numbers of unit module) is abelian
  $U(n>1)$ non abelian
  $|\det(U)| = 1$ because $\det(U^+) = \det(U)^*$
  $\det(U^+U) = |\det(U)|^2 = \det(\mathbb{1}) = 1$
- Special unitary group $SU(n)$ as above but with $\det(U) = +1$

- **Special unitary group** $SU(n)$ as above but with $\det(U) = +1$

$U(n) = SU(n) \otimes U(1)$ — $SU(n)$: not simple; $U(1)$: simple

$N = n^2 - 1$ dimension

- **Orthogonal group** $O(n)$ real orthogonal matrices $n \times n$ $O^T = O^{-1}$

$\det(O) = \pm 1$

$N = \frac{n(n-1)}{2}$ dimension

- **Special orthogonal group** $SO(n)$

$\det(O) = +1$

$N = \frac{n(n-1)}{2}$

[ex. $SO(3)$ is the rotation group

- Infinitesimal transformation

$g(d\alpha_a) = 1 + i t_a \alpha_a$ → generator

$g(\alpha_a = 0) = 1$ (no transformation)

$t_a = -i \left. \frac{dg}{d\alpha_a} \right|_{\alpha_a = 0}$

$g(\alpha_a) = \lim_{k \to \infty} \left(1 + i t_a \frac{\alpha_a}{k}\right)^k = e^{i t_a \alpha_a}$ (ex. L_x, L_y, L_z)

To have $e^{i\alpha_a t_a} e^{i\beta_b t_b} = e^{i\gamma_c t_c}$ one must have

$$[t_a, t_b] = i f_{abc} t_c$$

$a, b, c \in \{1, 2 \dots N\}$

→ structure constants of the group

N^3 real numbers which define the group

totally antisymmetric in (a, b, c)

e.g. $[L_i, L_j] = i \varepsilon_{ijk} L_k$ group $SU(2)$: $f_{abc} = \varepsilon_{abc}$

Jacobi identity:

$$[t_a, [t_b, t_c]] + [t_b, [t_c, t_a]] + [t_c, [t_a, t_b]] = 0$$

$$f_{abd} f_{cde} + f_{bcd} f_{ade} + f_{cad} f_{bde} = 0$$

Rank of group = number of commuting generators

e.g. $SU(2)$: dimension = 3 (L_x, L_y, L_z)

rank = 1 (e.g. L_z)

- **Representations** (= matrices)

  For $SU(n), O(n)$ the smallest matrices are $n \times n$ { fundamental representation

  e.g. $SU(2)$: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  but also

  3×3 $\Sigma_x = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\Sigma_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$ $\Sigma_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

  etc.

- **Conjugate representation**

  Given t_a $(a = 1 \dots N)$ one can define $\boxed{\bar{t}_a = -t_a^T}$ (same dimension)

  which also respect the commutation relations:

  $$[\bar{t}_a, \bar{t}_b] = [t_b, t_a]^T = -[t_a, t_b]^T = -(i f_{abc} t_c)^T$$

  $$= -i f_{abc} t_c^T = i f_{abc} (-t_c^T) = i f_{abc} \bar{t}_c$$

- **Adjoint representation**

  $N \times N$ matrices ↑ dimension of the group $\boxed{(t_a)_{ij} = -i f_{aij}}$ $a, i, j = 1, 2, \dots N$

- **Casimir operators**

  combinations of generators which commute with all generators (their representations are $\propto \mathbb{1}$)

  # of Casimir operators = rank of the group

  The simplest is $t_a t_a = t_1^2 + t_2^2 + \dots + t_N^2$

  ex. $SU(2)$ rank 1 : $\vec{L}^2 = L_x^2 + L_y + L_z^2$

  (can be used to classify states together with L_3)

☐ **Group $SU(n)$** dimension $N = n^2 - 1$

$\mathrm{Tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$ t_a are Hermitian

$C_2 \propto \mathbb{1}$:

$C_2 = c_2^{(F)} \mathbb{1}_{(n \times n)}$ Fundamental and conjugate representatio n

$c_2^{(F)} = \frac{n^2 - 1}{2n}$

$C_2 = c_2^{(A)} \mathbb{1}_{(N \times N)}$ Adjoint representation

$c_2^{(A)} = n$

$SU(2)$: dimension $N = 3$

rank $= 1$

$tr(t_a t_b) = \frac{1}{2}\delta_{ab}$

$[t_a, t_b] = i\varepsilon_{abc} t_c$

- Fundamental repres.: $t_a = \frac{\sigma_a}{2}$ ($a=1,2,3$) Pauli matrices 2x2
- Conjugate repres. = fundamental repres. (only for SU(2))

  because $\exists S : S\sigma_a^* S^{-1} = -\sigma_a$ $\quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  so that $SU^*S^{-1} = U$

  Can define $t_\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$

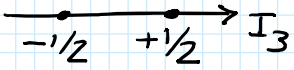
- Adjoint repres. 3x3 $\quad \Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\Sigma_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$ $\Sigma_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

  but also (change of basis) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

☐ Isospin is described by group SU(2) (like spin)

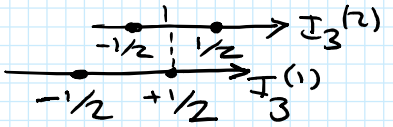
States are classified by (I, I_3), for each I along one axis

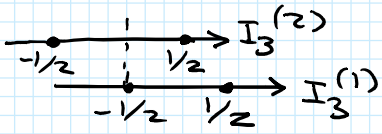
$I = 1/2$, $I_3 = \pm 1/2$



Combination of two $I = 1/2$ states:

put the above figure on top of each state





to obtain

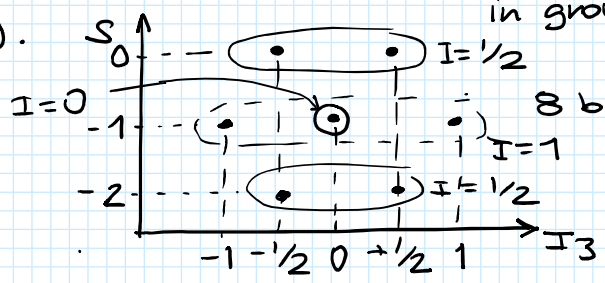


two states with $I_3 = 0$ ($I = 0$ and $I = 1$)

☐ Adding strangeness S

particles are classified along two axes (I_3, S) in groups of similar masses

e.g.



8 baryons, 10 baryons, 8 mesons etc.

- SU(3): dimension $N = 8$ ($a = 1, 2, \ldots 8$)

  rank 2

Fundamental representation 3x3 matrices $t_a = \frac{1}{2}\lambda_a$

rank 2

Fundamental representation 3×3 matrices $t_a = \frac{1}{2}\lambda_a$

Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longleftrightarrow I_3$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Longleftrightarrow S$$

Two diagonal (conventional choice) (rank 2)

$$tr(t_a t_b) = \frac{1}{2}\delta_{ab}$$

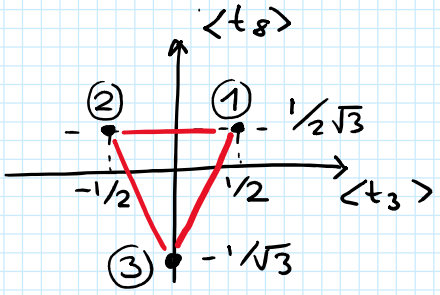
$$[t_a, t_b] = i f_{abc} t_c \qquad a, b, c = \{1 \dots 8\}$$

$\rightarrow f_{213} = -1$ etc.

$$\begin{cases} f_{123} = 1 \\ f_{147} = f_{246} = f_{257} = f_{345} = 1/2 \\ f_{156} = f_{367} = -1/2 \\ f_{458} = f_{678} = \sqrt{3}/2 \end{cases}$$

all others are 0

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



3 representation

$$\begin{cases} t_3|1\rangle = \frac{1}{2}|1\rangle \\ t_3|2\rangle = -\frac{1}{2}|2\rangle \\ t_3|3\rangle = 0 \end{cases}$$

← if $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ t_3 is I_3

$$\begin{cases} t_8|1\rangle = \frac{1}{2\sqrt{3}}|1\rangle \\ t_8|2\rangle = \frac{1}{2\sqrt{3}}|2\rangle \\ t_8|3\rangle = -\frac{1}{\sqrt{3}}|3\rangle \end{cases}$$

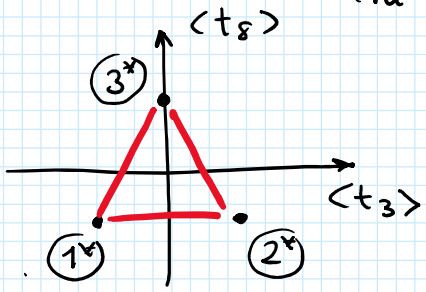
t_8 is $S + \frac{1}{2\sqrt{3}}$ if $S(s) = -1$

conjugate representation $|1^*\rangle, |2^*\rangle, |3^*\rangle$ with opposite values for I_3, S and transforming with

$$\overline{t_a} = -\lambda_a^T = -\lambda_a^*$$

⟨t8⟩

$$\bar{\lambda}_a = -\lambda_a^T = -\lambda_a^*$$



$\bar{3}$ representation

- Combining representations

$3 \otimes 3 = 6 \oplus 3$ in SU(3)

$3 \otimes 3 \otimes 3 = \dots$ 999

$3 \otimes \bar{3} = \dots$ $9\bar{9}$

Note that in SU(2)

$3 \otimes 3 = 5 \oplus 3 \oplus 1$ different

We get sets of particles with similar masses arranged in multiplets, but the mass differences are larger than between (p, n) or $(\pi^\pm, \pi^0)$ for isospin. The SU(3) symmetry is more approximate

- Can be extended to other quarks, SU(4), SU(5) etc. but symmetry is badly broken