FUNDAM. INT.

Wednesday, November 4, 2026 10:38 AM

(u, d, s) group $SU(3)$

$$\begin{cases} I_3 = \frac{1}{2}\lambda_3 = T_3 \\ Y = \frac{1}{\sqrt{3}}\lambda_8 = \frac{2}{\sqrt{3}}T_8 \end{cases}$$

I_3: 3rd component of isospin

Y: hypercharge

$T_a = \frac{1}{2}\lambda_a$ (λ_a: Gell-Mann matrices)

$Y = B + S$ ← strangeness

↑ baryon number

$B = +1$ baryons
$B = -1$ antibaryons
$B = 0$ other

charge $Q = I_3 + \frac{Y}{2}$

Gell-Mann – Nishijima

With more quarks: $Y = B + S + C + B' \pm T$ ← top, ← beauty, ← charm

Sign of strangeness: $S(s) = -1$
" " charm q.n.: $C(c) = +1$
" " beauty " : $B(b) = -1$
" " top " : $T(t) = +1$

(sign = sign of quark charge)

$Q = +2/3$: u c t
$Q = -1/3$: d s b

($Q = -2/3$: $\bar{u}$ $\bar{c}$ $\bar{t}$
$Q = +1/3$: $\bar{d}$ $\bar{s}$ $\bar{b}$)

$B(q) = +1/3$ $\quad B(\bar{q}) = -1/3$ so that

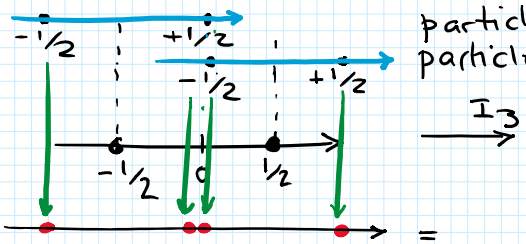
$B(qqq) = +1$
$B(q\bar{q}) = 0$
$B(\bar{q}\bar{q}\bar{q}) = -1$

Composition of multiplets (e.g. quarks)

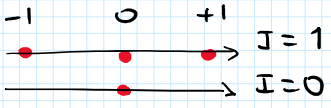
(1) Graphical approach:

Superimpose one diagram on the other centered on each state

- Ex. Two $I = 1/2$ states (doublets = fundamental representation) in $SU(2)$:

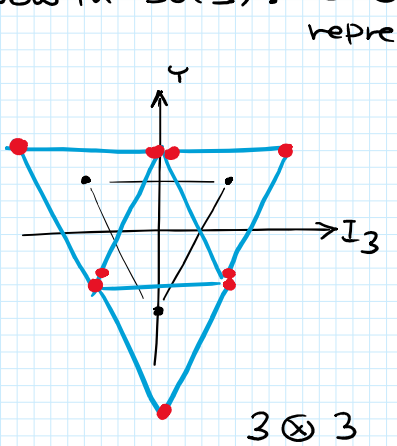


$=$

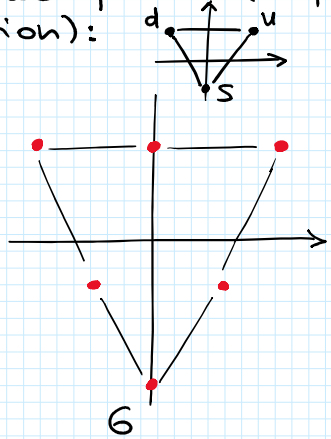


$2 \otimes 2 = 3 \oplus 1$

- Now in $SU(3)$: Combining two quarks (triplets = fundamental representation):



$=$



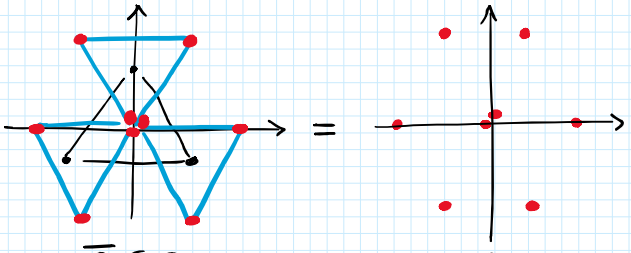
$3 \otimes 3 = 6 \oplus \bar{3}$

A different representation

$3 \otimes 3 = 6$ ⊕ $\bar{3}$

A different representation with 3 states, which is the same as for $\bar{q}$

Now $3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3$:



$\bar{3} \otimes 3 = 8 \oplus 1$

$6 \otimes 3 = \ldots$

(2) Young Tableaux approach :

Diagrams which describe the permutation symmetry of n objects

1 box □ for each particle, numbers to indicate states

arranged in rows for symmetric states

in columns for antisymmetric states

e.g. □ = 1 particle

 2 particles (symmetric state)

 2 particles (antisymmetric state) etc.

Proper tableau : { each row is no longer than the one above
each column is no longer than the one on the left

 NO  NO  YES

Standard arrangement : fill each box with numbers i ($1 \leq i \leq n$ for SU(n)) such that :

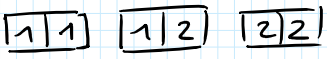
- numbers do not decrease along rows →
- numbers do increase along columns ↓

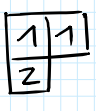
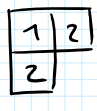
⇒ The number of different standard arrangements is the multiplicity

ex. SU(2) $i = 1,2$

□ = 2 ()

 = 1 ()

 = 3 ()

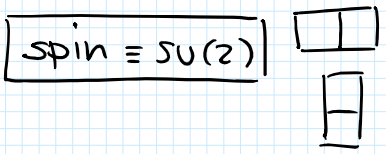
 = 2 ( )

example : two spin 1/2 particles $|\uparrow\rangle$ =  $|\downarrow\rangle$ =

spin ≡ SU(2)  = $\uparrow\uparrow$, $(\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}$, $\downarrow\downarrow$ (all symmetric)

(antisymmetric)

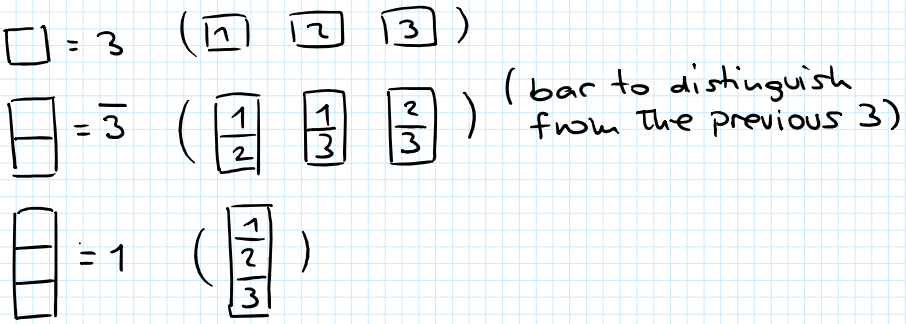
spin ≡ SU(2)



$= \uparrow\uparrow \;,\; (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2} \;,\; \downarrow\downarrow$ (all symmetric)

$= (\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$ (antisymmetric)

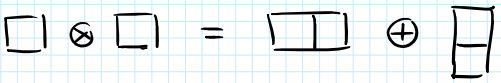
ex. SU(3) $i = 1,2,3$



Exercise:  = ?

 = ?

Combining representations:

- Just two boxes is easy : 

which means

In SU(2): $2 \otimes 2 = 3 \oplus 1$

In SU(3): $3 \otimes 3 = 6 \oplus \bar{3}$

- Three boxes ?

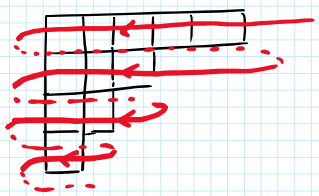


General algorithm to combine two diagrams:

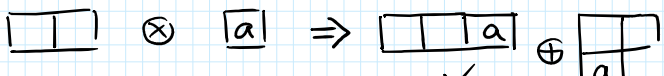
- draw the two diagrams
- in the one on the right put <u>letters</u> : a in first row
  b in second row
  etc.
- attach one lettered box to the other diagram, starting from the first in the first row, making sure it is <u>proper</u>, in all possible ways (no column with more than n boxes) with the same letter not appearing more than once in a column
- continue with next letter in first row ...
- continue with second row...

(if the same configuration is obtained more times just keep one)

- keep only diagrams in which, along the path in red, you never encounter at any time more b than a, more c than b etc.



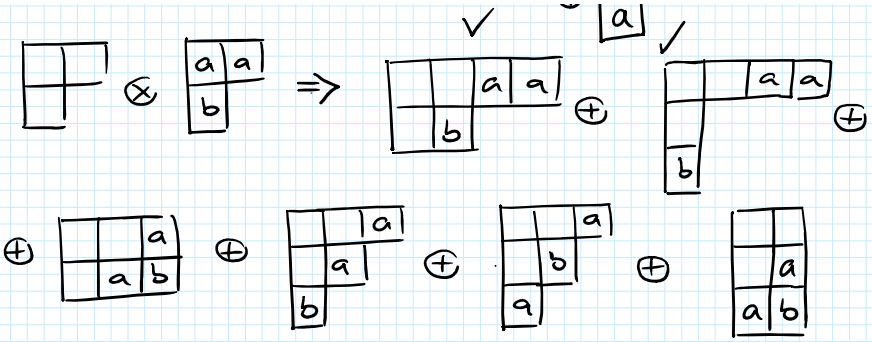
example:



example:



example:



Exercise: write the multiplicities for SU(3)

Exercise: what happens for the same combination in SU(2)?

Back to particles

- We study the symmetry SU(2) if two quarks (u,d) are not distinguished by strong interactions = isospin (good)
- We study the symmetry SU(3) if three quarks (u,d,s) are not distinguished by strong interactions (less good)
- With four, five etc. quarks the symmetries are bad...

- Let's consider only SU(3) and start with mesons

The fundamental representation is 3: $q = (u, d, s)$ ▽

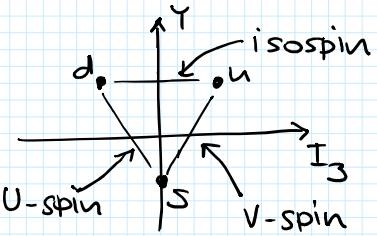
and the antiquarks are a different $\bar{3}$: $\bar{q} = (\bar{u}, \bar{d}, \bar{s})$ △

	I	I_3	S	B	Y	Q
u	1/2	+1/2	0	1/3	1/3	+2/3
d	1/2	-1/2	0	1/3	1/3	-1/3
s	0	0	-1	1/3	-2/3	-1/3

(opposite I_3, S, B for antiquarks)

SU(3) contains SU(2) of isospin as a subgroup, corresponding to the exchange $u \leftrightarrow d$, but also two other SU(2) corresponding to $d \leftrightarrow s$ ("U-spin") and $u \leftrightarrow s$ ("V-spin")

Ladder operators:
$$\begin{cases} I_\pm = T_1 \pm iT_2 \quad (I_3 = T_3) \\ U_\pm = T_6 \pm iT_7 \\ V_\pm = T_4 \pm iT_5 \end{cases}$$



	I_+	I_-	U_+	U_-	V_+	V_-
u	0	d	0	0	0	s
d	u	0	0	s	0	0
s	0	0	d	0	u	0
$\bar{u}$	$-\bar{d}$	0	0	0	$-\bar{s}$	0
$\bar{d}$	0	$-\bar{u}$	$-\bar{s}$	0	0	0
$\bar{s}$	0	0	0	$-\bar{d}$	0	$-\bar{u}$

Remember: for SU(2) we can use the same operators also for $\bar{q}$ if we add some minus signs:

Consider the state with the smallest I_3: $|d\bar{u}\rangle$ $(I_3 = -\frac{1}{2} - \frac{1}{2} = -1)$

$I_- |d\bar{u}\rangle = 0$ $\qquad |d\bar{u}\rangle = |1, -1\rangle$

Consider the state with the smallest I_3: $|d\bar{u}\rangle$ ($I_3 = -\frac{1}{2} - \frac{1}{2} = -1$)

$I_- |d\bar{u}\rangle = 0$ $\qquad |d\bar{u}\rangle = |1,-1\rangle$

$I_+ |d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle = \sqrt{2}\,|1,0\rangle$

$I_+ (|u\bar{u}\rangle - |d\bar{d}\rangle) = (0 - |u\bar{d}\rangle - |u\bar{d}\rangle + 0) = 2|1,+1\rangle$

so that

$$\begin{cases} |1,-1\rangle = |d\bar{u}\rangle & : |\pi^-\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) & : |\pi^0\rangle \\ |1,+1\rangle = -|u\bar{d}\rangle & : |\pi^+\rangle \end{cases}$$

The fourth combination is

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle)$$

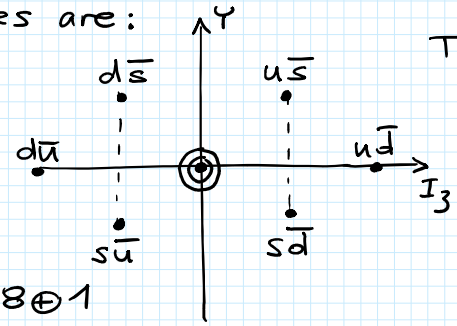
and $I_\pm |0,0\rangle = 0$

The same could have been done using $U_\pm$ or $V_\pm$, obtaining

$\frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |s\bar{s}\rangle)$ or $\frac{1}{\sqrt{2}}(|d\bar{d}\rangle - |s\bar{s}\rangle)$

which are not all independent from $\frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ (they add to $\emptyset$).

So the states are:



$3 \otimes \bar{3} = 8 \oplus 1$

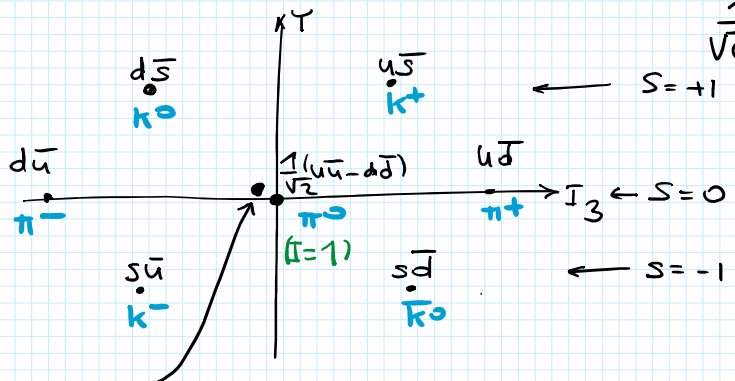
The states with $I_3 = 0 = Y$ are three:

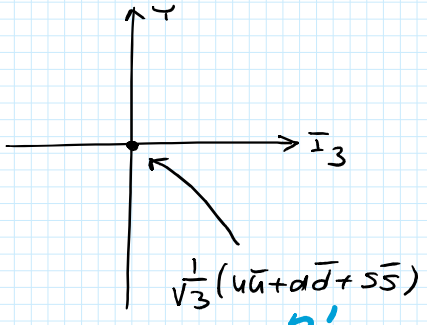
- $\frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ $\qquad (I=1)$
- one which must be in 1

  $|s\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$ $(I=0)$

  which has $I_\pm |s\rangle = 0$, $U_\pm |s\rangle = 0$, $V_\pm |s\rangle = 0$
- A third one orthogonal to the other two

  $\frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle)$ $(I=0)$





$\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})$

$\eta\,(I=0)$

If SU(3) symmetry is exact they should all be identical (for strong interactions)

This one could be different

$+1)=0$

8	$I=1$	π^-, π^0, π^+	$S=0$	~ 140 MeV	
	$I=0$	η	$S=0$	~ 550 MeV	
	$I=1/2$	K^0, K^+	$S=1$	~ 500 MeV	
	$I=1/2$	$K^-, \bar{K}^0$	$S=-1$		
1	$I=0$	η'	$S=0$	~ 960 MeV	

These are all pseudoscalar mesons : $J^P = 0^-$

- Recall : from $q\bar{q}$ of spin $1/2$ and flavour (u,d,s) :

$\psi(\vec{r}_1,\vec{r}_2)$ · $\chi(\uparrow,\downarrow)$ · $\Phi(u,d,s)$
space spin flavour

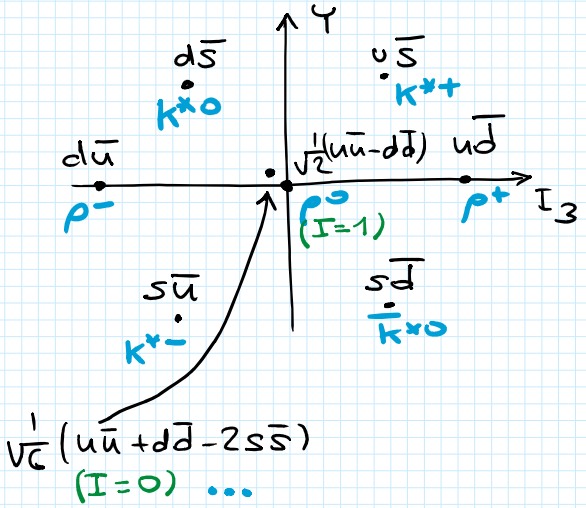
$Y_\ell^m(\theta,\varphi)$

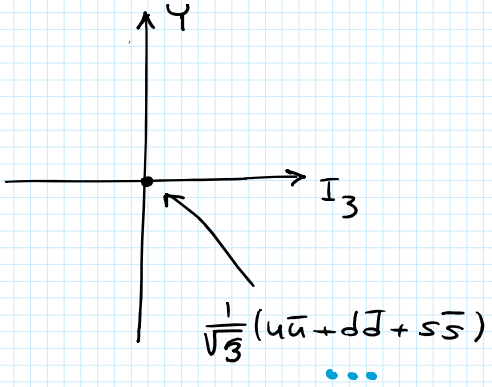
$L=0$ (lowest mass states) — $S=0$ or $S=1$ — what we just discussed

$$P(q\bar{q}) = (-1)^{L=0}\,(-1)^{S}\,\underbrace{P(q)P(\bar{q})}_{-1}$$

$$\begin{cases} L=0,\ S=0 \Rightarrow J=0 \\ P=(-1)^L \underbrace{P(q)P(\bar{q})}_{-1} = -1 \end{cases}$$
$+1$

- Those with $L=0$, $S=1 \Rightarrow J=1$ $\quad P=-1$ are the vector mesons $\quad J^P = 1^-$ with the same quark composition





- For each of those families there will be other states of higher mass corresponding to $L>0$