FUND. INT.

Wednesday, November 11, 2026 10:44 AM

Gauge theories

- Gauge invariance of EM:

$$\begin{cases} \phi \to \phi' = \phi - \dfrac{\partial \chi}{\partial t} \\ \vec{A} \to \vec{A}' = \vec{A} + \nabla \chi \end{cases} \quad \chi: \text{any function of } (\vec{x}, t)$$

$$A_\mu \to A'_\mu - \partial_\mu \chi \qquad \left(\phi, -\vec{A}\right) \quad \left(\partial/\partial t, +\nabla\right)$$

(gauge transfomation)

A_μ and A'_μ describe the same physics

- QM: $\psi \to \psi' = e^{i\alpha}\psi$ ($\to U(1)$) also called "gauge transfomation" is a symmetry ($|\psi|^2$ unchanged) for constant α

but $\psi \to \psi' = e^{iq\chi(x)}\psi$ local (space-time dependant) gauge transfomation

is it a symmetry?

Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \longrightarrow e^{iq\chi}\left[i\gamma^\mu \partial_\mu - q(\partial_\mu \chi) - m\right]\psi = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = q\gamma^\mu(\partial_\mu \chi)\psi \neq 0$$

Dirac equation is not left unchanged → not a symmetry

- EM can be described by using minimal coupling on mechanics

$$\left.\begin{aligned} E &\to E - q\phi \\ \vec{p} &\to \vec{p} - q\vec{A} \end{aligned}\right\} \quad p_\mu \to p_\mu - qA_\mu$$

which in QM where $\left.\begin{aligned} E &= i\partial/\partial t \\ \vec{p} &= -i\nabla \end{aligned}\right\}$ becomes $i\partial_\mu \to i\partial_\mu - qA_\mu$

$(i\gamma^\mu \partial_\mu - m)\psi = 0 \longrightarrow (i\gamma^\mu \partial_\mu - m)\psi = q\gamma^\mu A_\mu \psi$

Dirac eq. for an electron

Dirac eq. for an electron interacting with EM field $A_\mu(x)$

- Now note that applying both

- $\begin{cases} \psi \to \psi' = e^{iq\chi(x)}\psi \\ A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi(x) \end{cases}$ one has, for Dirac eq. for an interacting electron

$(i\gamma^\mu \partial_\mu - m)\psi = q\gamma^\mu A_\mu \psi \to$

$$e^{iq\chi}(i\gamma^\mu \partial_\mu - m)\psi = e^{iq\chi}\left[q\gamma^\mu(\partial_\mu \chi)\psi + q\gamma^\mu A_\mu \psi - q\gamma^\mu(\partial_\mu \chi)\psi\right]$$

the same equation.

Local gauge (phase) transfomations are a symmetry if ψ interacts with the EM field.

The requirement of local gauge symmetry imposes the existence of the EM field with the above interaction term. "gauge theory"

All our theories of fundamental interactions (apart gravity) are gauge theories.

- QCD is a more complicated version of EM in which the transfomation of ψ is not simply a change of phase (group $U(1)$) but a rotation in 3-dimensional colour space [group $SU(3)$]

8 colored gluons $J^P = 1^-$ instead of 1 uncharged photon $J^P = 1^-$

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$J^P = 1^-$ $J^P = 1^-$

$m_g = 0$ $m_\gamma = 0$

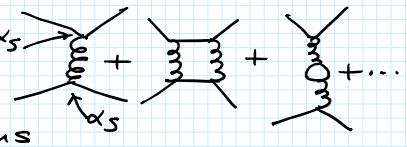
Problem: why $m_g = 0$ does not give a long-distance ($V \propto 1/r$) interaction? [Strong interactions are short-distance (V decreases much faster than $1/r$)]

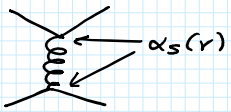
Because colour is confined and gluons (coloured) cannot move far

- The coupling constant is the adimensional number $\alpha_s = \frac{g^2}{4\pi\hbar c}$

(Just like $\alpha = \frac{e^2}{4\pi\hbar c}$)

but while $\alpha \simeq 1/137 \ll 1$, one has $\alpha_s \simeq 1$ difficult to use perturbation theory

The interaction of two electrons is well described by the lowest-order diagram  but the interaction of two quarks requires using many (infinite) diagrams 

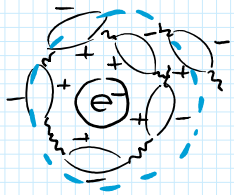
However the sum of the most important diagrams can be described simply as 

in which however the colour charge or α_s depends on distance r or squared momentum transfer q^2

(close interaction at short distance $\leftrightarrow$ large q^2 and viceversa

"Running coupling constant" $\alpha_s(q^2)$

- This happens (to a small extent) also in EM and can be visualized as the polarization of vacuum around a charge: virtual e^+e^- pairs are always present in vacuum, and e^+ are closer to an electron than e^-



Any sphere contains more e^+ than e^- and partially screens the charge of the central e^-: seen from distance the electron charge appears to be smaller $V(r) = \frac{q_1 q_2 |e|^2}{r}$

$q(r)$ decreases as r increases.

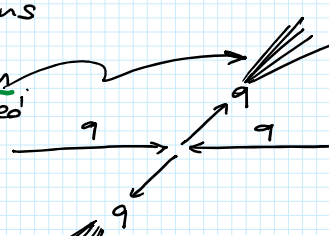
In QCD the same happens, but now there is also the effect of the virtual gluons which carry colour, and the net effect is opposite: the colour charge decreases at small distances

The interaction is stronger at large distances than at small ones

→ "Asymptotic freedom" quarks are free when very close together

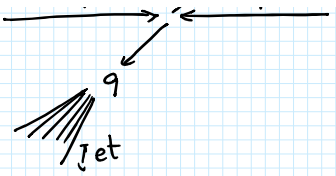
→ At large distances interaction energy becomes so large that new $q\bar{q}$ pairs are created and combine with the original quarks into colourless hadrons

hadronization cannot be described by theory in perturbative way ($\alpha_s \gtrsim 1$)



jet of hadrons (which still roughly follow the direction of the original quark)

by theory in perturbative way ($\alpha_s \gtrsim 1$)



- The strong interaction of hadrons (which have no colour) is the residual effect of the strong interaction, just as neutral atoms can still interact electromagnetically a little

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SUMMARY OF GROUPS USED:

SU(2) spin rotation symmetry	SU(2) isospin $(u \leftrightarrow d)$	SU(3) $(u \leftrightarrow d, s)$	SU(4)... $(u \leftrightarrow d, s, c)$	SU(3) colour symmetry
exact for all interactions	flavour symmetry			exact for all interactions (but only quarks and gluons have colour)
	very good	quite good for strong int.	poor	

Note an SU(3) flavour symmetry

The central states of the meson octets, composed of $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ should decompose into

$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ $\quad$ $\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$ $\quad$ $\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$

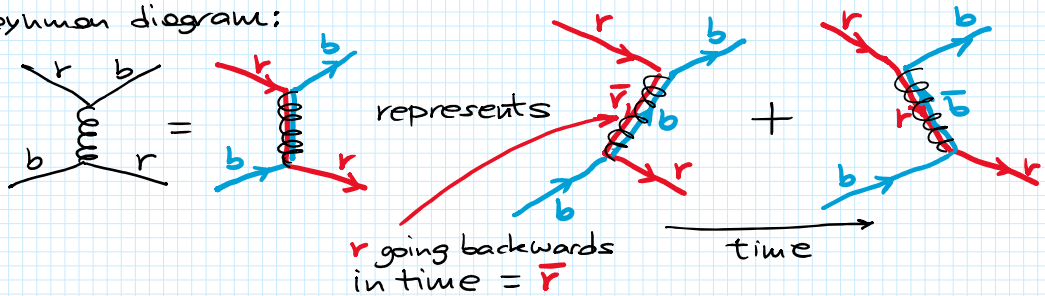
	octet $I=1$	octet $I=0$	singlet $I=0$
Pseudoscalar mesons:	π^0	η	η'
Vector mesons:	ρ^0	$\approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$ (ω)	$\approx s\bar{s}$ (ϕ)

mixtures of octet and singlet: $SU(3)_F$ symmetry not exact

Gluons $\quad J^P = 1^- \quad m = 0$

Colour is exactly conserved → colour lines never end but quarks do change colour by gluon interaction:

Feynman diagram:



So gluons carry one colour and one anticolour:

$r\bar{g}$ $\quad g\bar{r}$ $\quad r\bar{b}$ $\quad b\bar{r}$ $\quad g\bar{b}$ $\quad b\bar{g}$ $\quad r\bar{r}$ $\quad g\bar{g}$ $\quad b\bar{b}$

9 possibilities $(r, g, b) \times (\bar{r}, \bar{g}, \bar{b})$

But the three with no net colour $r\bar{r}$ $g\bar{g}$ $b\bar{b}$ actually combine in two coloured ones: $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$, $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

and a colourless one: $\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ the only one on which

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and a colourless one: $\boxed{\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})}$ the only one on which all raising and lowering operators give $\emptyset$ (singlet): $3\otimes\bar{3} = 8\oplus 1$

[Just like for spin $\uparrow\downarrow$ and $\downarrow\uparrow$ have $S_3=0$, but there are really two combinations $(\uparrow\downarrow+\downarrow\uparrow)/\sqrt{2}$ with $S=1$ and $(\uparrow\downarrow-\downarrow\uparrow)/\sqrt{2}$ with $S=0$]

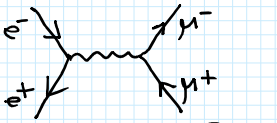
This one does not correspond to a gluon, all gluons have colour [8 gluons, not 9]

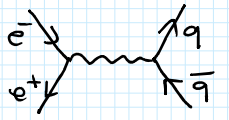
Experimental verification of colour existence

Consider $e^+e^- \to$ hadrons

which is just $e^+e^- \to q\bar{q} \to$ hadrons (hadronization, probability = 1)

The elementary process (EM) must be comparable

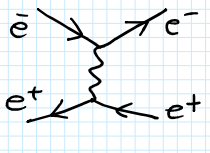
to $e^+e^- \to \mu^+\mu^-$ 



$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$

$\sigma = 4\pi\alpha^2$

except for the factor Q_q^2 ← charge of the quark (while $|Q_\mu|=1$)

[Note that $e^+e^- \to e^+e^-$ is different because it also has the diagram 

which is not possible in $e^+e^- \to q\bar{q}$ nor in $e^+e^- \to \mu^+\mu^-$]

So $$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{\text{quarks}} (Q_q)^2$$

depending on c.m. energy because if $\sqrt{s} < 2m_{q_i}$ $q_i\bar{q}_i$ cannot be produced

Measurement: apart from resonances, the value is constant and makes steps at some values of $\sqrt{s}$

		experiment:
$\sqrt{s} \lesssim 3$ GeV (only u,d,s):	$(2/3)^2_u + (-1/3)^2_d + (-1/3)^2_s = 2/3$	2
$3\text{ GeV} \lesssim \sqrt{s} \lesssim 9.5$ GeV (u,d,s,c):	same as ↑ $+(2/3)^2_c = 10/9$	$10/3$
$\sqrt{s} \gtrsim 9.5$ GeV (u,d,s,c,b):	same as ↑ $+(-1/3)^2_b = 11/9$	$11/3$

Experiment is 3x of above estimate:

indication that 3 time more final states are possible

because $q\bar{q}$ can occur with 3 colors $(r\bar{r}, g\bar{g}, b\bar{b})$

Number of colors = 3

The result is not exact: one should take into account that the denominator also includes $e^+e^- \to \mu^+\mu^-\gamma$ with an undetected low energy γ, and the numerator also $e^+e^- \to q\bar{q}\gamma$ and $e^+e^- \to q\bar{q}g$

the denominator also includes $e^+e^- \to \mu^+\mu^-\gamma$ with an undetected low energy γ, and the numerator also $e^+e^- \to q\bar{q}\gamma$ and $e^+e^- \to q\bar{q}g$ but taking also these into account theory matches experiment