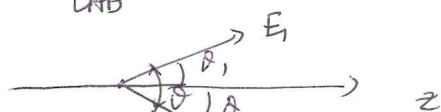
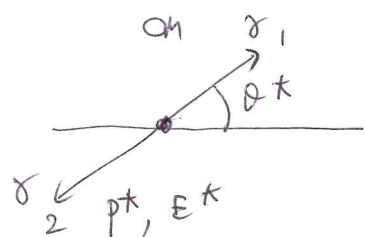


Exercise: in-flight decay $\pi^0 \rightarrow 2\gamma$ $E_\pi = 14 \text{ GeV}$

- Determine min/max energies in lab
- Determine min opening angle in lab



θ = angle between the two photons

General rule: try to avoid Lorentz transformations, and use invariants as much as possible.

Useful invariants: $(p_1 + p_2)^2$ and $P_{\text{Tot}}(P_i) = (P_1 + P_2)P_i$

For photons they are the same, $p_1^2 = p_2^2 = 0$

$$(P_{\text{Tot}})^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2p_1 \cdot p_2 = m_\pi^2 = s$$

Calculate in **CM**: $2p_1 p_2 = 2(p^{*2} + p^{*2} \cos \theta) = 4p^{*2}$

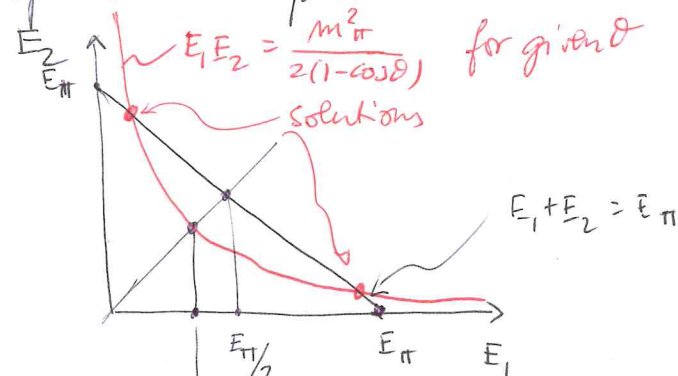
$$E^* = p^* = m_\pi/2$$

$$\begin{aligned} \text{Calculate in LAB: } 2p_1 p_2 &= 2(E_1 E_2 - E_1 E_2 \cos \theta) = \\ &= 2E_1 E_2 (1 - \cos \theta) \rightarrow E_1 E_2 = \frac{m_\pi^2}{2(1 - \cos \theta)} \end{aligned}$$

From energy conservation $E_1 + E_2 = E_\pi$

$$E_1, E_2 \text{ are the solution of } E^2 - E_\pi E + \frac{m_\pi^2}{2(1 - \cos \theta)} = 0$$

Graphical interpretation:



Varying θ , the hyperbola changes

$$\frac{m_\pi}{2} \leq \frac{m_\pi}{\sqrt{2(1 - \cos \theta)}} \quad (\text{obtained when } E_1 = E_2 \rightarrow E_1^2 = E_2^2 = \frac{m_\pi^2}{2(1 - \cos \theta)})$$

$$E_1, E_2 = \frac{1}{2} \left(E_\pi \pm \sqrt{E_\pi^2 - \frac{2m_\pi^2}{1 - \cos \theta}} \right)$$

For back to back one has $E_{\text{min}}/E_{\text{max}} \rightarrow \cos \theta = -1$

(because $(1 - \cos \theta)$ is max, $\frac{1}{1 - \cos \theta}$ is min, $\sqrt{E_\pi^2 - \frac{2m_\pi^2}{1 - \cos \theta}}$ is max)

$$E_{\text{min, max}} = \frac{1}{2} \left(E_\pi \pm \sqrt{E_\pi^2 - \frac{2m_\pi^2}{1 - \cos \theta}} \right) = \frac{E_\pi \pm p_\pi}{2}$$

$$\begin{aligned} \text{For Ultra Rel. } \pi: E_{\text{max}} &\approx E_\pi = 14 \text{ GeV} / E_{\text{min}} = \frac{E_\pi - p_\pi}{2} = \frac{E_\pi}{2} (1 - \beta_\pi) = \frac{E_\pi}{2} \frac{1 - \beta_\pi}{1 + \beta_\pi} \\ &\approx \frac{E_\pi}{4\gamma_\pi^2} = \frac{m_\pi}{4\gamma_\pi} = 0.35 \text{ MeV} \\ &\quad \beta_\pi \approx 1, \gamma_\pi = 100 \quad (\text{also from: } E_{\text{min}} E_{\text{max}} = \frac{m_\pi^2}{4} \rightarrow E_{\text{min}} = \frac{m_\pi^2}{4E_\pi} = \frac{m_\pi}{4\gamma_\pi}) \end{aligned}$$

Min angle when $\Delta = 0$ (hyperbola tangent to line)

$$E_\pi^2 = \frac{2m_\pi^2}{1 - \cos \theta_{\text{min}}} \rightarrow 1 - \cos \theta_{\text{min}} = \frac{2}{\gamma_\pi^2} \rightarrow \frac{\theta_{\text{min}}^2}{2} \approx \frac{2}{\gamma_\pi^2} \rightarrow \theta_{\text{min}} \approx \frac{2}{\gamma_\pi} = 20 \text{ mrad}$$

Using Lorentz transformations:

$$E_1 = \gamma_\pi (p^* + \beta_\pi p^* \cos \theta^*) = \gamma_\pi p^* (1 + \beta_\pi \cos \theta^*) \rightarrow \text{Since } d^N/d\cos \theta^* = \text{const, also } \frac{d^N}{dE_1} = \text{constant,}$$

• Calculation of individual angles θ_1, θ_2

Also to calculate individual angles one can use invariants

$$P_{TOT} \cdot P_1 = (P_1 + P_2) \cdot P_1 = P_1^2 + P_1 P_2 = P_1 P_2 = m_\pi^2/2$$

In the LAB frame $P_{TOT} = (E_\pi, \vec{p}_\pi)$ $P_1 = (E_1, \vec{p}_1)$

$$P_{TOT} \cdot P_1 = E_\pi \cdot E_1 - p_\pi p_1 \cos \theta_1 = E_1 E_\pi (1 - \beta_\pi \cos \theta_1)$$

with $\beta_\pi = p_\pi/E_\pi$. From the two expressions:

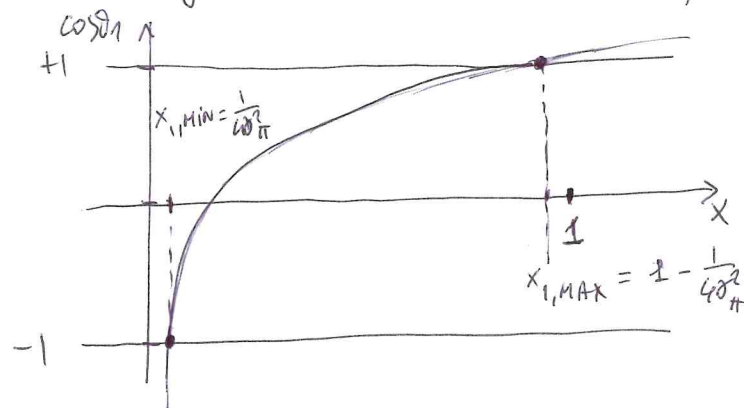
$$E_1/E_\pi = \frac{m_\pi^2}{2 E_\pi^2 (1 - \beta_\pi \cos \theta_1)} = \frac{1}{2 \gamma_\pi^2 (1 - \beta_\pi \cos \theta_1)} = x_1$$

Let's define $x_1 = E_1/E_\pi$ $x_2 = E_2/E_\pi$ $x_1 + x_2 = 1$

Limits: $\theta_1 = \pi$ $x_{1, MIN} = \frac{1}{4 \gamma_\pi^2}$ (consistent with previous page)

$$\theta_1 = 0 \quad x_{1, MAX} = \frac{1}{2 \gamma_\pi^2 (1 - \beta_\pi)} = \frac{1}{2} \frac{1 - \beta_\pi^2}{1 - \beta_\pi} = \frac{1 + \beta_\pi}{2} \approx 1$$

• Inverting the relation: $\cos \theta_1 = \frac{1}{\beta_\pi} \left(1 - \frac{1}{2 x_1 \gamma_\pi^2} \right)$



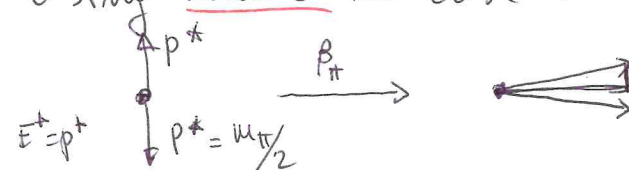
The minimum opening angle is obtained in the symmetric configuration $x_1 = x_2 = 1/2$ $\cos \theta_1 = \frac{1}{\beta_\pi} \left(1 - \frac{1}{\gamma_\pi^2} \right)$

Temptation: one could say that $\beta_\pi \approx 1$, but in this formula it does not work because $\cos \theta_1$ is also ≈ 1 and so you have differences like $1 - \beta_\pi$, which is not zero:

$$\cos \theta_1 = \frac{1}{\beta_\pi} \left(1 - 1 + \beta_\pi^2 \right) = \beta_\pi \rightarrow 1 - \frac{\theta_1^2}{2} = \beta_\pi \quad \theta_1^2 = 2(1 - \beta_\pi)$$

For UR: $1 - \beta_\pi = \frac{1 - \beta_\pi^2}{1 + \beta_\pi} = \frac{1}{2 \gamma_\pi^2} \rightarrow \theta_1^2 = \frac{1}{\gamma_\pi^2}$

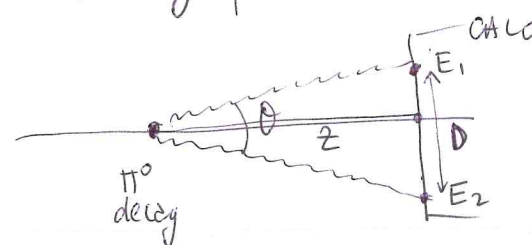
• Using Lorentz in core $\theta^* = \pi/2$



$$\begin{aligned} p_y &= p^* = m_\pi/2 \\ p_z &= \gamma_\pi (0 + \beta_\pi p^*) = \gamma_\pi \beta_\pi p^* \\ E_1 &= \gamma_\pi (p^* + \beta_\pi \cdot 0) = \gamma_\pi p^* \\ \theta_1 &= \frac{1}{\gamma_\pi} \end{aligned}$$

• Decay point of π

If one measures the Energy and position of the two photons with a calorimeter, one can determine the π^0 decay position:



$$E_1 E_2 = \frac{m_\pi^2}{2(1 - \cos \theta)} \rightarrow \cos \theta = 1 - \frac{m_\pi^2}{2 E_1 E_2}$$

$$\theta^2 \approx \frac{m_\pi^2}{E_1 E_2} \rightarrow \frac{D^2}{z^2} = \frac{m_\pi^2}{E_1 E_2}$$

$$z = D \sqrt{\frac{E_1 E_2}{m_\pi^2}}$$